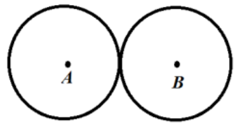
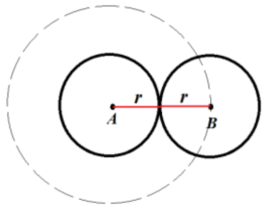
Example 1: A penny *A* is rolling around a second penny *B* without slipping until it returns to its starting point. How many revolutions does *penny A* make?



Solution: Two revolutions.

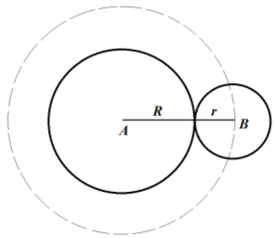
The distance *D* traveled by the centre of the circle *A* can be used as a representative distance traveled by the circle *A*.



$D = 2\pi(r + r) = 4\pi r$.

The number of revolutions is $\frac{4\pi r}{2\pi r} = 2$.

Theorem 1: Circle *A* is rolling around a second circle *B* without slipping until it returns to its starting point. The number of revolutions the circle *A* make is

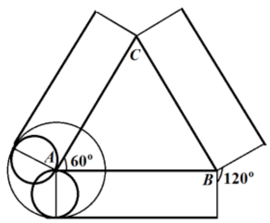


$$N = \frac{2\pi(R+r)}{2\pi \times r} = \frac{R}{r} + 1$$

The total distance the center of the circle travelled is
$D = 2\pi(R+r)$

The distance the center of the circle travelled when circle *B* travels one revolution is
$d = \frac{2\pi r}{R} \times (R+r)$.

Example 2: The side of equilateral ΔABC has length 2π. A circle with radius 1 rolls around the outside of Δ*ABC*. When the circle first returns to its original position, how many revolutions does it roll?



Solution:

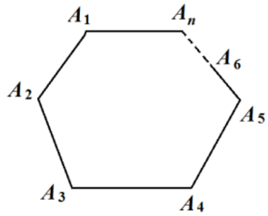
The circle rolls a 180° – 60° = 120° arc from point *A* of *AC* to point *A* on *AB* (1/3 of the circumference of the circle). The circle rolls one revolution from point *A* of *AB* to point *B* on *AB*.

Total it rolls 4 revolutions.

Theorem 2: Circle A is rolling around a regular n sides polygon with the side length the same as the circumference of the circle without slipping until it returns to its starting point. The number of revolutions the circle A make is $N = n + 1$.

Theorem 3.1: Circle A is rolling around a convex polygon without slipping until it returns to its starting point. If the length of the perimeter of the polygon is n times of the length of the circumference of the circle, the number of revolutions the circle A make is $N = n + 1$.

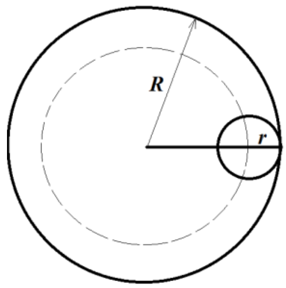
The circle rolls a $(180° - \angle A_1) + (180° - \angle A_2) + \ldots + (180° - \angle A_n)$ $= 180° \times n - (\angle A_1 + \angle A_2 + \ldots + \angle A_n) = 180° \times n - 180° \times (n - 2)$ $= 360°$ arc along all vertices.



Theorem 3.2: The distance the center of the circle travels is the sum of the length of the perimeter of the polygon and the length of the circumference of the circle.

Example 3: Circle A of radius 1 is rolling around inside a second circle B of radius 4 without slipping until it returns to its starting point. Find the number of revolutions the circle A makes.

Solution:



The distance D traveled by the centre of the small can be used as a representative distance traveled by it.

$D = 2\pi(R - r) = = 2\pi(4 - 1) = 6\pi$

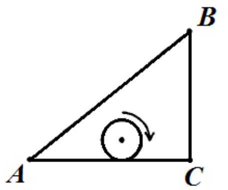
The number of revolutions is $\dfrac{6\pi}{2\pi \times 1} = 3$

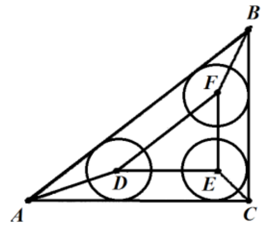
Theorem 4: Circle A with radius r is rolling around inside a second circle B with radius R without slipping until it returns to its starting point. The number of revolutions the circle A makes is $N = \dfrac{2\pi(R-r)}{2\pi \times r} = \dfrac{R}{r} - 1$.

Theorem 5: A circle with radius r is rolling around inside a triangle with sides a, b, and c without slipping until it returns to its starting point. The distance travelled by the center of the circle is:

The perimeter of ΔABC – the perimeter of the similar triangle ΔDEF $= a + b + c - m(a + b + c) = (1 - m)(a + b + c)$

where $m = 1 - \frac{pr}{A}$, and $p = \frac{1}{2}(a + b + c)$





Below shows how we got m:

We know that ΔDEF, of the center of the rolling circle, is similar to ΔABC, so we label its sides ma, mb, mc, for some m and $1 > m > 0$.

The area of ΔABC is $A = \sqrt{p(p-a)(p-b)(p-c)}$ (1)

where $p = \frac{1}{2}(a + b + c)$

The area of ΔDEF is $A_1 = \sqrt{mp(mp - ma)(mp - mb)(mp - mc)} = m^2 A$ (2)

Partition ΔABC into three trapezoids of altitude r and ΔDEF, and compute the area of ΔABC in terms of r:

$[ABC] = [DACE] + [CEFB] + [BFDA] + [DEF]$

$= \frac{1}{2}(r)(mb + b) + \frac{1}{2}(r)(ma + a) + \frac{1}{2}(r)(mc + c) + m^2 A$

$= \frac{1}{2}(r)(mb + b + ma + a + mc + c) + A = pr(m+1) + m^2 A$ (3)

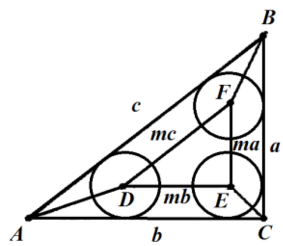
We know that (1) = (3). So we have

$A = pr(m+1) + m^2 A \quad \Rightarrow \quad pr(m+1) + m^2 A - A = 0$

$\Rightarrow \quad pr(m+1) + A(m+1)(m-1) = 0$.

Since $m \neq 0$, $m + 1 \neq 0$. Thus $pr + A(m-1) = 0 \quad \Rightarrow$

$m = 1 - \frac{pr}{A}$.



PROBLEMS

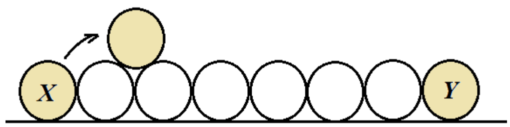
Problem 1: A small circle of radius 2 cm is rotating without slipping around a larger circle of radius 10 cm. When the circle first returns to its original position, how many revolutions does the circle roll?

Problem 2: The side of a regular hexagon has length 2π. A circle with radius 1 rolls around the outside of it. When the circle first returns to its original position, how many revolutions does the circle roll?

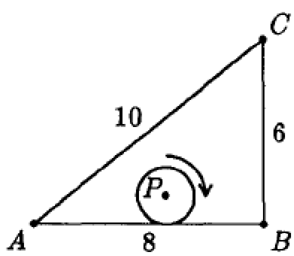
Problem 3: Circle *A* with radius 2 is rolling around inside a second circle *B* with radius 8 without slipping until it returns to its starting point. Find the number of revolutions the circle *A* makes.

Problem 4: A small circle of radius 2 cm is rotating without slipping around a larger circle of radius 9 cm. If the small circle starts with point *A* on its circumference in contact with the larger circle, find the exact distance traveled by the centre of the small circle before the point *A* next comes in contact with the large circle.

Problem 5: (UNL Probe I, 1993) A circle is rolled without slipping, across the top of the other six identical circles to get from the initial position x to the final position y. What is the number of revolutions it must make?



Problem 6: (1993 AMC 12) The sides of ΔABC have lengths 6, 8 and 10. A circle with center *P* and radius 1 rolls around the inside of ΔABC, always remaining tangent to at least one side of the triangle. When *P* first returns to its original position, through what distance has *P* traveled?



SOLUTIONS:

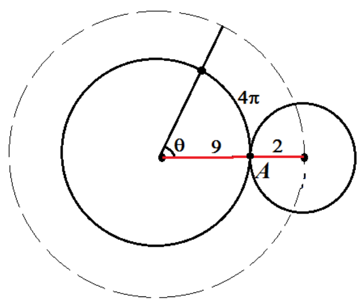
Problem 1: Solution:
By **Theorem 1,** the number of revolutions the circle *A* make is

$$N=\frac{2\pi(R+r)}{2\pi\times r}=\frac{R}{r}+1=\frac{10}{2}+1=6.$$

Problem 2: Solution:
By **Theorem 2**, the number of revolutions the circle *A* make is $N = n + 1 = 6 + 1 = 7$.

Problem 3: Solution:
By **Theorem 4,** the number of revolutions the circle A makes is $N = \frac{R}{r}-1=\frac{8}{2}-1=4-1=3$.

Problem 4: Solution:
Method 1:

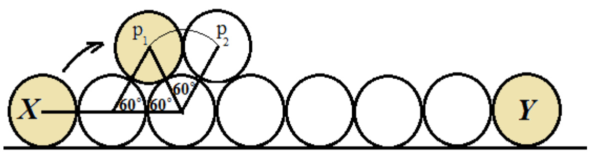


Since the circumference of the small circle is 4π, the central angle in the large circle between successive points of contact of the point A with the large circle is $\theta = s_1/r = 4\pi/9$.
The radius of the circle followed by the centre of the small circle is 11. Thus, the distance traveled by the centre of the small circle before the point A next comes in contact with the large circle is $s_2 = R\theta = (9 + 2)(4\pi/9) = 44\pi/9$.

Method 2:

By the Theorem 1, $d=\frac{2\pi r}{R}\times(R+r)=\frac{2\pi\times 2}{9}\times(9+2)=\frac{44\pi}{9}$.

Problem 5: Solution:



As a circle rolls through one revolution, its center travels a distance equal to the circumference. As the circle moves from position *X* to position p_1, the center moves a distance $2\pi/3\ (2r) = (4\pi r)/3$. As the circle moves from position p_1 to p_2, the center moves $\pi/3\ (2r) = (2\pi r)/3$. The total distance traveled by the center of the rolling circle is $2(4\pi r)/3 + 4(2\pi r)/3 = (16\pi r)/3$. Since each revolution is $2\pi r$, the number of revolutions is $16\pi r/3$ divided by $2\pi r$ which is 8/3.

Problem 6: Solution: 12
We solve this problem by two different methods other than the official solution.

Method 1:

By **Theorem 5:**

The distance = $a + b + c - m(a + b + c) = (1 - m)(a + b + c)$.

$$m = 1 - \frac{pr}{A} = 1 - \frac{\frac{10+8+6}{2}\times 1}{\frac{8\times 6}{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

So the distance = $(1 - m)(a + b + c) = (a + b + c)/2 = 12$.

Method 2:

We have $\tan\theta = \frac{y}{8}$. By the angle bisector Theorem, we get $\frac{8}{y} = \frac{10}{6-y} \Rightarrow y = \frac{8}{3}$

$$\tan\theta = \frac{r}{m} = \frac{1}{m} \quad \text{so} \quad \frac{1}{m} = \frac{\frac{8}{3}}{8} \Rightarrow m = 3$$

$$\tan\alpha = \frac{r}{n} = \frac{1}{n} \quad \text{and} \quad \frac{1}{n} = \frac{3}{6} \Rightarrow n = 2.$$

Thus the distance travelled = $8 - (m + 1) + 6 - (n + 1) + 10 - (m + n) = 8 - (3 + 1) + 6 - (2 + 1) + 10 - (3 + 2) = 12$.