Example 1: We have a two-pan balance scale comes with two weights: 1 gram and 2 grams. (a) If these weights can only be put in one pan during a weighing, how many different weighable values are there? (b) If these weights can be put in in either or both pans during a weighing, how many different weighable values are there?

Solution:

(a) If we only use one weight during each measurement, we are able to measure things weighing 1 gram or 2 grams. If we put both 1 gram and 2 grams weights together in one pan, we can measure things with 1 + 2 = 3 grams. So we get 3 weighable values: 1, 2, and 3.

(b) We know from (a) that we can get 3 weighable values: 1, 2, and 3. If we put 1 gram weight in one pan and 2 grams weight in another pan, we can measure things weighting 2 – 1 = 1 gram which we already have. So we still have 3 weighable values: 1, 2, and 3.

Example 2: We have a two-pan balance scale comes with two weights: 1 gram and 3 grams. (a) If these weights can only be put in one pan during a weighing, how many different weighable values are there? (b) If these weights can be put in either or both pans during a weighing, how many different weighable values are there?

Solution:

(a) If we only use one weight during each measurement, we are able to measure things weighing 1 gram or 3 grams. If we put both 1 gram and 3 grams weights together in one pan, we can measure things with 1 + 3 = 4 grams. So we get 3 weighable values: 1, 3, and 4.

(b) We know from (a) that we can get 3 weighable values: 1, 3, and 4. If we put 1 gram weight in one pan and 3 grams weight in another pan, we are able to measure things weighting 2 grams. So we have 4 weighable values: 1, 2, 3, and 4.

Example 3: We have a two-pan balance scale comes with two weights: 1 gram and 4 grams. (a) If these weights can only be put in one pan during a weighing, how many different weighable values are there? (b) If these weights can be put in in either or both pans during a weighing, how many different weighable values are there?

Solution:

(a) If we only use one weight during each measurement, we are able to measure things weighing 1 gram or 4 grams. If we put both 1 gram and 4 grams weights together in one pan, we can measure things with 1 + 4 = 5 grams. So we get 3 weighable values: 1, 4, and 5.

(b) We know from (a) that we can get 3 weighable values: 1, 4, and 5. If we put 1 gram weight in one pan and 4 grams weight in another pan, we are able to measure things weighting 3 gram. So we have 4 weighable values: 1, 3, 4, and 5.

Example 4: We have a two-pan balance scale comes with three weights: 1 gram, 3 grams and 9 grams. (a) If these weights can only be put in one pan during a weighing, how many different weighable values are there? (b) If these weights can be put in in either or both pans during a weighing, how many different weighable values are there?

Solution:

(a) If we only use one weight during each measurement, we are able to measure things weighing 1 gram, 3 grams, or 9 grams. If we use two weights each time, we get 3 weighable values (1 + 3 = 4, 1 + 9 = 10, and 3 + 9 = 12). If we use three weights at one time, we get one more weighable value (1 + 3 + 9 = 13). Or in other way:

$$\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=3+3+1=7.$$

(b) We know from (a) that we can get 7 weighable values: 1, 3, 9, 4, 10, 12, and 13.

If we put 1 gram weight in one pan and 3 grams weight in another pan, we are able to measure things weighting 2 grams.

If we put 1 gram weight in one pan and 9 grams weight in another pan, we are able to measure things weighting 8 grams.

If we put 3 gram weight in one pan and 9 grams weight in another pan, we are able to measure things weighting 6 grams.

If we put 1 gram weight in one pan and 3 and 9 grams weights in another pan, we are able to measure things weighting 11 grams.

If we put 3 grams weight in one pan and 1 and 9 grams weights in another pan, we are able to measure things weighting 7 grams.

If we put 9 grams weight in one pan and 1 and 3 grams weights in another pan, we are able to measure things weighting 5 grams.

So we have 7 + 6 = 13 weighable values: 1 up to 13.

THEOREM 1: There is *n* number of different weights.

(a). If the weights can only be put to one side of the scale, the greatest number of weighable values is $2^n - 1$.

n	1	2	3	4	5	6 …. n
greatest number of weighable values:	1	3	7	15	31	63…., $2^n - 1$.

(b). If the weights are $2^0, 2^1, 2^2, 2^3, \ldots, 2^n - 1$, the weighable values are from 1 to $2^n - 1$.

THEOREM 2: There is *n* number of different weights.

(a). If the weights can be put in both sides of the scale, the greatest number of weighable values is $\frac{3^n - 1}{2}$.

n	1	2	3	4	5 …. n
greatest number of weighable values	1	4	13	40	121 …. $\frac{3^n - 1}{2}$.

(b). If the weights are $3^0, 3^1, 3^2, 3^3, \ldots, 3^n - 1$, the weighable values are from 1 to $\frac{3^n - 1}{2}$.

Problem 1: There are five weighing cubes of 1g, 2g, 4g, 8g, and 16g. How many different values can be weighted if these weights can only be put in one side of the scale during a weighing?

Problem 2: There are five weighing cubes of 1g, 2g, 4g, 8g, and 16g. How many different values can be weighted if these weights can be put in both sides of the scale during a weighing?

Problem 3: (2005 Mathcounts Handbook) A two-pan balance scale comes with a collection of weights. Each weight weighs a whole number of grams. Weights can be put in only one pan during a weighing. To ensure any whole number of grams up to 60 grams can be measured, what is the minimum number of weights needed in the collection?

Problem 4: A two-pan balance scale comes with a collection of weights. Each weight weighs a whole number of grams. Weights can be put in either or both pans during a weighing. To ensure any whole number of grams up to 100 grams can be measured, what is the minimum number of weights needed in the collection?

SOLUTIONS

Problem 1: Solution:
Method 1: By the theorem 1 (a), we know the number of different values is $2^n - 1 = 2^5 - 1 = 31$.

Method 2: Get one, two,… five cubes each time, the resulting sums are different. So know the number of different values is $\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5} = 2^5 - 1 = 31$.

Problem 2: Solution:
By the theorem 1 (a), we know the number of different values is $2^n - 1 = 2^5 - 1 = 31$ if these weights can only be put in side of the scale during a weighing.

We see that the greatest weighable value is 1 + 2 + 4 + 8 + 16 = 31 and the smallest weighable value is 1. There are 31 whole numbers from 1 to 31. So we still get 31 weighable values even we are allowed to put the weights in both sides of the scale during a weighing.

Problem 3: Solution: 6 weights.
The 6 weights are 1g, 2g, 4g, 8g, 16g, and 32g.

These 6 weights can measure any weight from 1g to up to $2^n - 1 = 2^6 - 1 = 64 - 1 = 63$g.

Problem 4: Solution: 5 weights.
The 5 weights are 1g, 3g, 9g, 27g, and 81g.
These 5 weights can measure any weight from 1g to up to $\frac{3^n - 1}{2} = 242/2 = 121$ g.