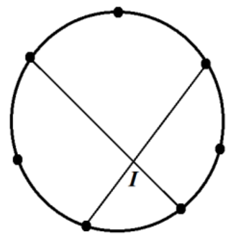
Example 1: (2003 Mathcounts Handbook) Seven points are distributed around a circle. All possible chords, using the given seven oints as endpoints, are drawn. What is the greatest possible number of points of intersection inside the circle? The point I is an example of one such point.

Solution:



The intersection points are determined by 4 points on the circle. The chords of a group of 4 points intersect only at one point. Then one intersection point corresponds to 4 points on the circle.
The greatest possible number of points of intersection inside the circle is $\binom{7}{4} = 35$.

Theorem 1: The greatest number of intersection points of *n* lines on a plane is $\binom{n}{2}$.

The intersection points are determined by 2 lines. Two lines will intersect at one point.

Theorem 2: The greatest number of intersection points of all diagonals of a convex *n*-gon is $\binom{n}{4}$.

The intersection points are determined by 4 vertices on the circle. The diagonals of a group of 4 vertices intersect only at one point.

Theorem 3: The greatest number of intersection points of all possible chords drawn from *n* points on a circle is $\binom{n}{4}$.

The intersection points are determined by 4 points on the circle. The chords of a group of 4 points intersect only at one point. Then one intersection point corresponds to 4 points on the circle.

Example 2: (Mathcounts) What is the maximum of points of intersection when 5 lines intersect each other?

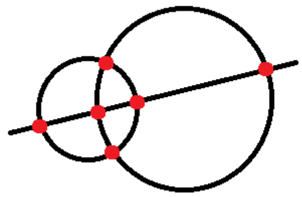
Solution: 10 (points).

Two lines will intersect at one point. The greatest number of intersection points of *n* lines on a plane is $\binom{n}{2} = \binom{5}{2} = 10$.

Example 3: (Mathcounts) What is maximum number of intersection points in a diagram with two circles of unequal diameter and a straight line?

Solution: 6 (points)

We get six points:



Example 4: (Mathcounts Handbooks) Three distinct lines in a plane may intersect in *k* points. What are all possible values of *k*?

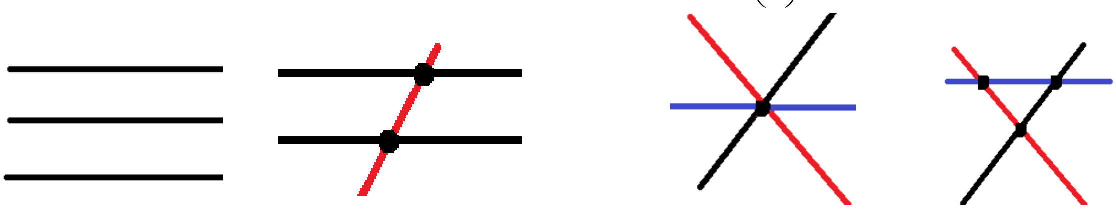
Solution: 0, 1, 2, and 3

If three lines are parallel, they intersect at 0 point.

If two of the three lines are parallel, they intersect at 2 points.

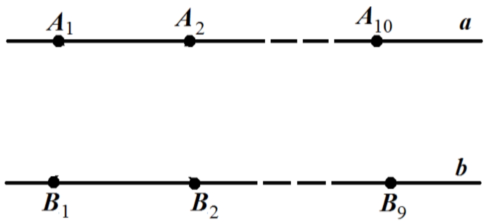
If three lines are concurrent (they pass through one point), they intersect at 1 point.

If no three of the five lines are concurrent, they intersect at $\binom{3}{2} = 3$ points.



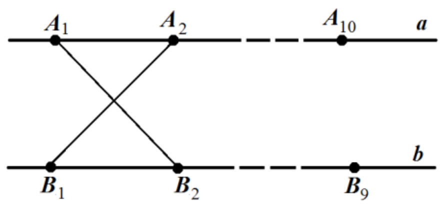
Thepossible values are 0, 1, 2, and 3.

Example 5: As shown in the figure, line *a* is parallel to line *b*. There are 10 points on line a and nine points on line b. If we connect each point on a with each point on b, we get many lines. If no three of these line intersects at one point, what is the number of intersection points bertween *a* and *b*?



Solution:

We get one intersection point by connecting two points on *a* with two points on *b*.



We have $\binom{10}{2}$ ways to select two points from 10 points on *a*, and $\binom{9}{2}$ ways to select two points from 9 points on *b*.

So the number of intersection points is $\binom{10}{2} \times \binom{9}{2} = 45 \times 36 = 1620$.

PROBLEMS

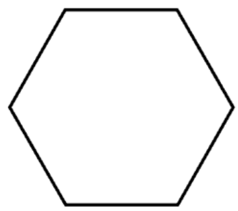
Problem 1: What is the maximum number of intersection points inside a circle possible by all the chords formed by connecting six points on the circle?

Problem 2: (Mathcounts) What is the maximum number of intersection points possible for ten non-collinear lines in the same plane?

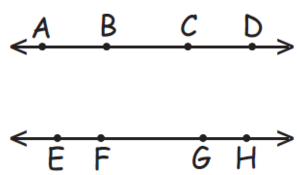
Problem 3: (Mathcounts) What is the largest number of points of intersection of a square, a circle and a line?

Problem 4: (Mathcounts Competitions) Five straight lines may lie in a plane in many different ways. For example, they may all intersect at a single point. Thus, one is a possitble number of intersection points for the five lines. Find the sum of all the whole numbers from 0 to 10 which cannot represent the total number of intersections of five distinct co-planar lines.

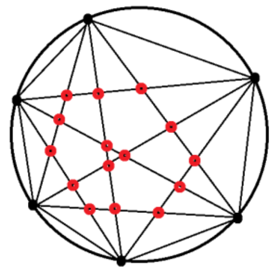
Problem 5: If all the diaonals of a regular hexagon are drawn, what is the greatest possible number of points of intersection inside the hexagon?



Problem 6: (2009 Mathcounts Handbook) Connect each point labeled on line *AD* with each point labeled on line *EH*. What is the total number of points of intersection between lines *AD* and *EH* that are created by the newly drawn segments? (Note: No three segments intersect at one point between the lines *AD* and *EH*.)



SOLUTIONS

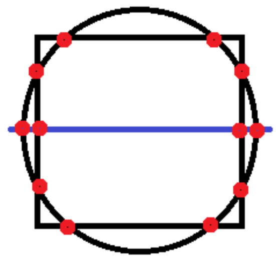


Problem 1: Solution: 15.

Method 1:
We count and we get 15 points.

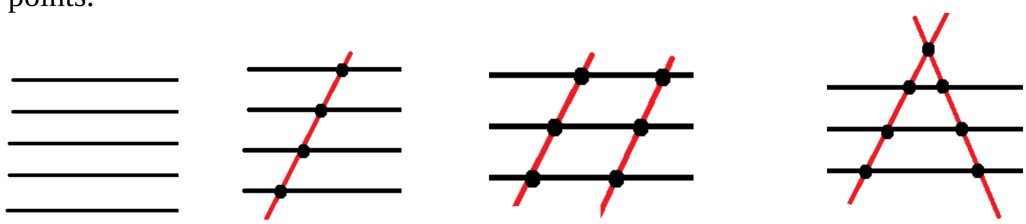
Method 2:
The greatest number of intersection points of all possible chords drawn from n points on a circle is $\binom{n}{4} = \binom{6}{4} = 15$.

Problem 2: Solution: 45.
Two lines will intersect at one point. The greatest number of intersection points of n lines on a plane is $\binom{n}{2} = \binom{10}{2} = 45$.



Problem 3: Solution: 12.
One circle and one square can intersect at most 8 points. When one line is joined, it can intersect two points with the circle and two points with the square. Total we get 8 + 2 + 2 = 12 points.

Problem 4: Solution: 5.
If five lines are parallel, they intersect at 0 point.
If four of the five lines are parallel, they intersect at 4 points.
If three of the five lines are parallel, and other two lines are parallel, they intersect at 6 points.
If three of the five lines are parallel, and other two lines are not parallel, they intersect at 7 points.

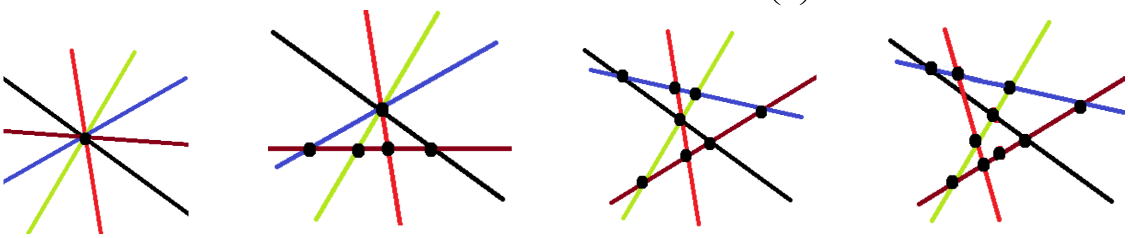


If five lines are concurrent (they pass through one point), they intersect at 1 point.

If four of the five lines are concurrent, they intersect at $\binom{5}{2}-\binom{4}{2}+1=5$ points.

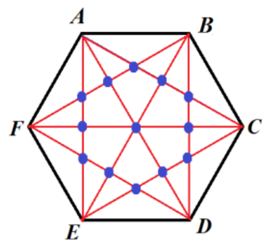
If three of the five lines are concurrent, they intersect at $\binom{5}{2}-\binom{3}{2}+1=8$ points.

If no three of the five lines are concurrent, they intersect at $\binom{5}{2}=10$ points.

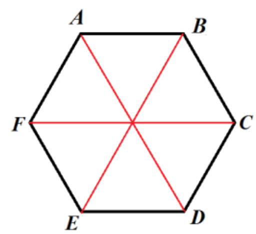


The numbers of intersetion points not achieveable are 2 and 3. The sum of them is 5.

Problem 5: Solution: 13.
Method 1:
We draw and count. We get 13 points.



Method 2:
The intersection points are determined by 4 vertices on the circle. The diagonals of a group of 4 vertices intersect only at one point.
The greatest number of intersection points of all diagonals of a convex n-gon is $\binom{n}{4}=\binom{6}{4}=15$.



However, Four points of *AB* – *DE*, *BC* – *EF*, and *AC* – *DF* will produce only one point of intersection.
So the answer is 15 – 3 + 1 = 13.

Problem 6: Solution: 36.
We get one intersection point by connecting two points on the top line with two points on the bottom line. We have $\binom{4}{2}$ ways to select two points from *A*, *B*, *C*, *D*, and $\binom{4}{2}$ ways to select two points from *E*, *F*, *G*, *H*.

So the number of intersection points is $\binom{4}{2}\times\binom{4}{2}=6\times 6=36$.