Example 1: I have a frog who hops stairs either one or two steps at a time. Our staircase has 11 steps. How many different ways can the frog hop the stairs?

Solution: 144.

The frog has one way to hop up to stair 1.

It has two ways to hop up to stair 2 (it can hop one stair at a time twice or it can hop two stairs at a time).

The frog has three ways to hop up to stair 3 (it can hop one stair at a time three times, 1 + 1 + 1; or it can hop one stair first then hop two stairs at a time, 1 + 2; or it can hop two stairs at a time first then hop one stair, 2 + 1).

The pattern is as follows:

Stairs	Number of ways	Note
6	13	(5 + 8 = 13)
5	8	(3 + 5 = 8)
4	5	(2 + 3 = 5)
3	3	(1 + 1 + 1, 1 + 2, and 2 + 1)
2	2	(1 + 1, and 2)
1	1	1

Please note that this is the Fibonacci numbers:

Stairs:	1	2	3	4	5	6	7	8	9	10	11
Ways:	1	2	3	5	8	13	21	34	55	89	**144.**

GENERAL CASE:

N_1 means that the number of ways the frog can hop up to the stair 1
N_2 means that the number of ways the frog can hop up to the stair 2
N_3 means that the number of ways the frog can hop up to the stair 3
N_4 means that the number of ways the frog can hop up to the stair 4
N_5 means that the number of ways the frog can hop up to the stair 5
N_6 means that the number of ways the frog can hop up to the stair 6.

Case 1: The frog can hop 1 stair or 2 stairs at a time
We need to get N_1 and N_2. We call them "the initial conditions". $N_1 = 1$, and $N_2 = 2$.

With N_1 and N_2, we are able to calculate any term:
$N_3 = N_2 + N_1$.
$N_4 = N_3 + N_2$.
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Case 2: The frog can hop 1, 2, or 3 stairs at a time.

We need to get N_1, N_2 and N_3. $N_1 = 1$, $N_2 = 2$, and $N_3 = 4$.

With N_1, N_2 and N_3, we are able to calculate any term:
$N_4 = N_3 + N_2 + N_1$.
$N_5 = N_4 + N_3 + N_2$.
$N_6 = N_5 + N_4 + N_3$.
…………

Case 3: The frog can hop 1, 2, 3, or 4 stairs at a time.

We need to get N_1, N_2 , N_3, and N_4. $N_1 = 1$, $N_2 = 2$, $N_3 = 4$, and $N_4 = 8$.

With N_1, N_2 , N_3, and N_4, we are able to calculate any term:
$N_5 = N_4 + N_3 + N_2 + N_1$
$N_6 = N_5 + N_4 + N_3 + N_2$
…………

Case 4: The frog can hop 1, or 3 stairs at a time.

We see that the order of 1, 2, 3 is interrupted: 1, $\boxed{2}$, 3.

We still need to get N_1, N_2 , and N_3. But we notice that $\boxed{2}$ is missing. $N_1 = 1$, $N_2 = 1$, and $N_3 = 2$.

With N_1, N_2 , and N_3, we are able to calculate any term with a little modification of the case 2:

$N_4 = N_3 + \boxed{N_2} + N_1 = N_3 + N_1$

$N_5 = N_4 + \boxed{N_3} + N_2 = N_4 + N_2$

…………

Case 5: The frog can hop 2, or 3 stairs at a time.

We see that the order of 1, 2, 3 is interrupted: $\boxed{1}$ 2, 3.

We still need to get N_1, N_2 , and N_3. But we notice that $\boxed{1}$ is missing. $N_1 = 0$, $N_2 = 1$, and $N_3 = 1$.

With N_1, N_2, and N_3, we are able to calculate any term with a little modification of the case 2:

$N_4 = \boxed{N_3} + N_2 + N_1 = N_2 + N_1$

$N_5 = \boxed{N_4} + N_3 + N_2 = N_3 + N_2$

…………

Case 6: The frog can hop 1, or 4 stairs at a time.

We see that the order of 1, 2, 3, 4 is interrupted: 1, $\boxed{2}$, $\boxed{3}$, 4.

We still need to get N_1, N_2, N_3. and N_4. But we notice that $\boxed{2}$ and $\boxed{3}$ are missing. $N_1 = 1$, $N_2 = 1$, $N_3 = 1$, and $N_4 = 2$.

With N_1, N_2, N_3. and N_4, we are able to calculate any term with a little modification of the case 3:

$N_5 = N_4 + \boxed{N_3} + \boxed{N_2} + N_1 = N_4 + N_1$

$N_6 = N_5 + \boxed{N_4} + \boxed{N_3} + N_2 = N_5 + N_2$

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Use the method outlined here you are able to generate as many cases as you like. We just summarize 11 commonly used cases below:

CASES COMMONLY USED:

Stairs hopped	Forms	Number of ways	Cases
1, 2	(1. 2)	$N_3 = N_2 + N_1$	(1)
1, 2, 3	(1, 2, 3)	$N_4 = N_3 + N_2 + N_1$	(2)
1, 2, 3, 4	(1, 2, 3, 4)	$N_5 = N_4 + N_3 + N_2 + N_1$	(3)
1, 3	(1, □, 3)	$N_4 = N_3 + 0 + N_1 = N_3 + N_1$	(4)
2, 3	(□, 2, 3)	$N_4 = 0 + N_2 + N_1 = N_2 + N_1$	(5)
1, 4	(1, □, □, 4)	$N_5 = N_4 + 0 + 0 + N_1 = N_4 + N_1$	(6)
2, 4	(□, 2, □, 4)	$N_5 = 0 + N_3 + 0 + N_1 = N_3 + N_1$	(7)
3, 4	(□, 2, □, 4)	$N_5 = 0 + 0 + N_2 + N_1 = N_2 + N_1$	(8)
1, 2, 4	(1 2, □, 4)	$N_5 = N_4 + N_3 + 0 + N_1 = N_4 + N_3 + N_1$	(9)
1, 3, 4	(1, □, 3, 4)	$N_5 = N_4 + 0 + N_2 + N_1 = N_4 + N_2 + N_1$	(10)
2, 3, 4	(□, 2, 3, 4)	$N_5 = 0 + N_3 + N_2 + N_1 = N_3 + N_2 + N_1$	(11)

Example 2: I have a pet cat Tiger who climbs stairs either one or three steps at a time. Our staircase has 11 steps. How many different ways can Tiger climb the stairs?

Solution: 41.

This is the case 4. We still need to get N_1, N_2 , and N_3. $N_4 = N_3 + N_1$.
Tiger has one way to climb stair 1, 1 way to climb stair 2, and 2 ways to climb stair 3 (1 + 1 + 1, and 3).
With the formula $N_4 = N_3 + N_1$, the sequence is as follows: 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, **41.**

Example 3: How many different ways to climb 10 stairs if one can only climb stairs two or three steps at a time?

Solution: 7.

This is the case 5. We need to get N_1, N_2 , and N_3. $N_4 = N_2 + N_1$
We have 0 way to climb stair 1, one way to climb up to stair 2, and one way to climb up to stair 3. With the formula $N_4 = N_2 + N_1$, the sequence can be obtained as follows:
0, 1, 1, 1, 2, 2, 3, 4, 5, **7**.

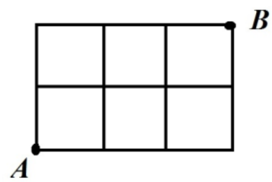
Example 4: (2010 Mathcounts Handbook) A frog is going to hop up the stairs from the first floor to the second floor. Hopping up the stairs one step at a time will require a total of ten hops. If the frog can hop one, two or three steps at a time, how many different sequences of hops are possible for the frog to reach the second floor with each hop being upward?

Solution: 274.

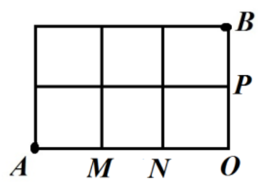
This is the case 2. We need to get N_1, N_2, and N_3. $N_4 = N_3 + N_2 + N_1$.
We have 1 way to climb stair 1, 2 ways to climb up to stair 2 (1 +1, or 2), and 4 ways to climb up to stair 3 (1 + 1+ 1, 1 + 2, 2 + 1, 3). With the formula $N_4 = N_3 + N_2 + N_1$, the sequence can be obtained as follows: 1, 2, 4, 7, 13, 24, 44, 81, 149, **274.**

Example 5: How many ways from A to B with the shortest distance walked if you can walk one step or two steps at a time. Figure shows that there are total 5 steps from *A* to *B*.

Solution: 80.



Step 1: Count the number of routes from *A* to *B*
So we have $\binom{5}{2} = 10$ routes



Step 2: Count the number of ways for each route.
This is the case 1. We need to get N_1 and N_2. $N_3 = N_2 + N_1$
The sequence is as follows: 1, 2, 3, 5, **8**.

Step 3: Multiply them together to get the solution: $10 \times 8 = 80$ ways.

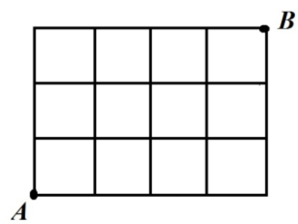
PROBLEMS

Problem 1: You can climb stairs with each step consisting of either 1 stair or 2 stairs. How many different ways are there to climb 10 stairs if the order of the steps is considered?

Problem 2: You have enough 1¢, 2¢, 3¢, and 4¢ stamps. You want to stick them in a row. How many ways are there to get a total of 8¢?

Problem 3: You have enough 2¢, 3¢, and 4 ¢ stamps and you want to stick them in a row. How many ways are there to get a total of 10¢?

Problem 4: How many ways from *A* to *B* with the shortest distance walked if you can walk one step or two steps at a time. Figure shows that there are total 5 steps from *A* to *B*.



SOLUTIONS:

Problem 1: Solution: 89.

This is the case 1. We need to get N_1 and N_2. $N_3 = N_2 + N_1$

The sequence is as follows: 1, 2, 3, 5, 8, 13, 21, 34, 55, **89.**

Problem 2: Solution: 108.

This is the case 3. We need to get N_1, N_2 , N_3, and N_4.

Stairs	# of ways	Note
4	8	(1 + 1 + 1 + 1, 1 + 1 +2, 1 + 3, 2 + 2, 4).
3	4	(1 + 1 + 1, 1 + 2, 2 + 1, 3)
2	2	(1 + 1, or 2)
1	1	1

With the formula $N_5 = N_4 + N_3 + N_2 + N_1$**,** the sequence can be obtained as follows: 1, 2, 4, 8, 15, 29, 56, **108.**

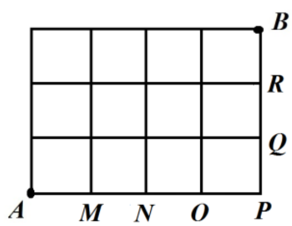
Problem 3: Solution: 17.

This is the case 11. We need to get N_1, N_2 , N_3, and N_4.

Stairs	# of ways	Note
4	2	(2 + 2 or 4)
3	1	(3)
2	1	(2)
1	0	

With the formula $N_5 = N_3 + N_2 + N_1$, the sequence can be obtained as follows: 0, 1, 1, 2, 2, 4, 5, 8, 11, **17**.

Problem 4: Solution: 735.



Step 1: Count the number of routes from A to B

So we have $\binom{7}{3} = 35$ routes

Step 2: Count the number of ways for each route.

This is the case 1. We need to get N_1 and N_2. $N_3 = N_2 + N_1$

The sequence is as follows: 1, 2, 3, 5, 8, 13, **21.**

Step 3: Multiply them together to get the solution: 35 × 21 = 735 ways.