1. WARM UP PROBLEMS

Try the following problems to see how many you are able to solve. If you have trouble to solve some, read the following section to learn some skills. Then come back to working on these problems. You will find that you are in a better position to solve these problems.

1. If p, r, and s are three different prime numbers greater than 2, and $n = p^2 \times r \times s$, how many positive factors, including 1 and n, does n have?

2. If one of the positive factors of 80 is to be chosen at random, what is the probability that the chosen factor will be a multiple of 10?

(A) $\frac{3}{5}$
(B) $\frac{5}{7}$
(C) $\frac{1}{5}$
(D) $\frac{2}{5}$
(E) $\frac{7}{8}$

3. A positive integer is said to be "quadruple-factorable" if it is the product of four consecutive integers. How many positive integers less than 10, 000 are quadruple -factorable?

$x^y = 8192$

4. In the equation above, x and y are positive integers. What is the greatest possible value of $x - y$?

5. How many positive integers less than 20 have exactly 2 factors?

(A) 20
(B) 10
(C) 9
(D) 8
(E) 7

6. How many of the positive integers from 1 to 100 have an odd number of factors?

(A) 10
(B) 9
(C) 8
(D) 5
(E) 3

7. How many even positive integral factors does 6006 have?

8. How many positive integral factors does N have if $N = 6^2 \times 15$?

9. If $n = 2^3 \times 3^2 \times 5$, how many odd positive factors does n have?

(A) 12
(B) 6
(C) 4
(D) 8
(E) 9

10. What is the smallest positive integer by which 252 can be multiplied so that result would be a perfect cube?

(A) 2352
(B) 98
(C) 294
(D) 196
(E) 49

11. How many factors of 21,600 are perfect squares?

(A) 24
(B) 20
(C) 16
(D) 14
(E) 12

12. How many perfect cube factors does $2^4 \times 3^6 \times 5^{10}$ have?

(A) 24
(B) 20
(C) 16
(D) 14
(E) 12

13. How many natural numbers n will make $\frac{36}{n-1}$ a natural number?

(A) 2
(B) 4
(C) 6
(D) 8
(E) 9

2. BASIC KNOWLEDGE

2.1. Terms

FACTORS: Factors are the numbers you multiply together to get the product.

$1\times 12 = 2 \times 6 = 3 \times 4 = 12 \quad \Rightarrow \quad$ 1, 2, 3, 4, 6 and 12 are factors of 12.

DIVISOR: A divisor is a number which divides the given number without leaving any remainder.

$\frac{6}{1} = 6 \qquad \frac{6}{2} = 3 \qquad \frac{6}{3} = 2 \qquad \frac{6}{6} = 1$

The denominators 1, 2, 3, and 6 are called the divisors of the numerator 6.

Note: A factor of the number is the same as a divisor of the number.

MULTIPLE: Multiples are the products when you multiply two or more numbers.

$24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6 \quad \Rightarrow$ 24 is a multiple of 1, 2, 3, 4, 6, 8, 12, and 24.

PRIME FACTORIZATIONS: Prime factorization of a number is to express the number as a product of factors that are all prime.

$12 = 2 \times 6 = 2 \times 2 \times 3 = 2^2 \times 3.$ $\qquad 1001 = 7 \times 11 \times 13.$

2.2. Fundamental theorem of arithmetic:

Any composite number, besides 0 and 1, can be written as a product of prime numbers.
And this expression is unique.

$36 = 6 \times 6 = (2 \times 3) \times (2 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 4 \times 9 = (2 \times 2) \times (3 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 3 \times 12 = 3 \times (2 \times 6) = 3 \times (2 \times 2 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 2 \times 18 = 2 \times (2 \times 9) = 2 \times (2 \times 3 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 1 \times 36 = 1 \times (2 \times 3)^2 = 2^2 \times 3^2$

2.3. Methods of prime factorization

The divisibility rules and the square root rule can be used to find the prime factorization of a number.

Example 1: What is the largest prime factor of 189?

Solution: 7.

Method 1: Since the sum of the digits is divisible by 3, 189 is divisible by 3.
$189 = 3 \times 63 = 3 \times 3 \times 21 = = 3 \times 3 \times 3 \times 7 = 3^3 \times 7$.

Note that 189 is also divisible by 7.

Method 2: Since $13^2 = 169 < 189$ and $14^2 = 196 > 189$, the square root of 189 is between 13 and 14.

So we use the prime numbers 13, 11, 7, and 3 to test and we get: $189 = 7 \times 27$.
Note that we do not use the prime number 5 in our test because we know that 189 is not divisible by 5.

2.4. Number of divisors

For an integer n greater than 1, let the prime factorization of n be

$$n = p_1^a p_2^b p_3^c \ldots p_k^m$$

Where $a, b, c, \ldots$, and m are nonnegative integers, $p_1, p_2, \ldots, p_k$ are prime numbers.

The number of divisors is:

$$d(n) = (a+1)(b+1)(c+1)\ldots(m+1)$$

Example 2: How many factors does $2^3 \cdot 3^6 \cdot 5$ have?

Solution: 56 factors.

The number of factors is $(3 + 1)(6 + 1)(1 + 1) = 56$.

Example 3: How many distinct positive integral factors would the following product have: (12)(15)(17)?

Solution: 36.

We do prime factorization of (12)(15)(17):
$(12)(15)(17) = 189 = 3 \times 2^2 \times 3 \times 5 \times 17 = 2^2 \times 3^2 \times 5^1 \times 17^1$
The number of divisors $d = (2 + 1)(2 + 1)(1 + 1)(1 +1) = 36$.

Example 4: How many positive integral factors does N have if $N = 6^2 \cdot 15$?

Solution: 24.

We do prime factorization first: $N = 6^2 \cdot 15 = (2 \cdot 3)^2 \cdot 3 \cdot 5 = 2^2 \cdot 3^3 \cdot 5$
The number of divisors $d = (2 + 1)(3 + 1)(1 + 1) = 24$.

3. PROBLEM SOLVING SKILLS

3.1 Find the number of even divisors

Skill: (1) Prime factorization of the given number. (2) Take out one 2. (3) Calculate the number of factors of the remaining number.

Theorem: The number of factors of a prime number is even (2).

Example 5: How many positive integer factors of 72 are even?

Solution: 9 (factors).
$72 = 3^2 \times 2^3 = 2(3^2 \times 2^2)$
Any factors of $3^2 \times 2$ will be a multiple of 2.
The number of positive integer factors of 72 are also multiples of 2 is: $(2 + 1)(2 + 1) = 9$.

Example 6: How many positive integer factors of 72 are also multiples of 4?

Solution: 6.

$72 = 3^2 \times 2^3 = 4(3^2 \times 2)$.
The number of positive integer factors of 72 that are also multiples of 3 is: $(2 + 1)(1 + 1) = 6$.

3.2 Find the number of odd divisors

Skill: (1) Prime factorization of the given number. (2) Take out all 2's. (3) Calculate the number of factors of the remaining number.

Theorem: The number of factors of a square number is odd.

Example 7: How many odd positive integers are factors of 100?

Solution: 3.
$100 = 5^2 \times 2^2 = 2^2(5^2)$
The number of odd factors of $2^2(5^2)$ is the same as the number of factors of (5^2) which is 3.

Example 8: How many of the positive integers from 1 to 10000 do not have an odd number of factors?

Solution: 9900.

Any square number will have an odd number of factors. There are $\sqrt{10000} = 100$ square numbers from 1 to 10000. So the answer is $10000 - 100 = 9900$.

3.3 Find the number of divisors that are the multiple of *m*

Steps: (1) Prime factorization of the given number. (2) Take out one m. (3) Calculate the number of factors of the remaining number.

Example 9: How many positive integer factors of 56 are also multiples of 4?

Solution: 4 (integers).

$56 = 7 \times 2^3 = 4(7 \times 2)$.

The number of factors of 7×2^3 which are also multiples of 4 is the same as the number of factors of (7×2) which is 4.

Example 10: How many positive integer factors of 56 are also multiples of 14?

Solution: 4 (integers).

$56 = 7 \times 2^3 = 14(2^2)$.

The number of factors of 2^2 is 3.

3.4 Find the number of divisors that are square numbers

Steps: (1) Prime factorization of the given number. (2) Group all integers with an even exponent and write them in the form of N^2. (3) Take out all integers left over. (4) Calculate the number of factors of N.

Example 11: The prime factorization of a certain number is $2^2 \cdot 3^2 \cdot 5$. How many of its positive integral factors are perfect squares?

Solution: 4.
$2^2 \cdot 3^2 \cdot 5 = (2 \cdot 3)^2 \cdot 5$.
Any factors of $(2 \cdot 3)$ will be a factor of perfect square.
The answer is $(1 + 1)(1 + 1) = 4$.

Example 12: How many of the positive integer factors of 432 are perfect squares? (Mathcounts Handbooks)

Solution: 6.

$432 = 2^4 \cdot 3^3 = (2^2 \cdot 3)^2 \cdot 3$.
Any factors of $(2^2 \cdot 3)$ will be a factor of perfect square.
The answer is $(2 + 1)(1 + 1) = 6$.

Example 13: How many odd perfect square factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?

Solution: 120.

We just look at the number of perfect square factors for $3^6 \times 5^{10} \times 7^9$.
$3^6 \times 5^{10} \times 7^9 = (3^3 \times 5^5 \times 7^4)^2 \times 7$.
There are (3 + 1)(5 + 1)(4 + 1) = 120 odd perfect square factors.

Example 14: How many of the positive integers from 1 to 100 have an odd number of factors?

Solution: 10.

Any square number will have an odd number of factors. There are $\sqrt{100} = 10$ square numbers from 1 to 100. So the answer is 10.

3.5 Find the number of divisors that are cubic numbers

Steps: (1) Prime factorization of the given number. (2) Group all integers with an odd exponent that is a multiple of 3 and write them in the form of N^3. (3) Take out all integers left over. (4) Calculate the number of factors of *N*.

Example 15: How many perfect cube factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?

Solution: 96.
$2^4 \times 3^6 \times 5^{10} \times 7^9 = (2^3 \times 3^6 \times 5^9 \times 7^9) \times 2^1 \times 5^1 = (2^1 \times 3^2 \times 5^3 \times 7^3)^3 \times 2^1 \times 5^1$
$\Rightarrow \quad (2^1 \times 3^2 \times 5^3 \times 7^3)^3 \qquad \Rightarrow \quad N = 2^1 \times 3^2 \times 5^3 \times 7^3$
$d(N) =$ (1 + 1) (2 + 1)(3 + 1)(3 + 1) = 2 × 3 × 4 × 4 = 96

Example 16: What is the least positive integer by which you could multiply 180 to get a product that is a perfect cube?

Solution: 150.

Let m^3 be the perfect cube and n be the smallest positive integer.
$180 = 2^2 \cdot 3^2 \cdot 5$.
$180 \times n = 2^2 \cdot 3^2 \cdot 5 \times n = m^3$.
$n = 2 \cdot 3 \cdot 5^2 = 150$.

4. EXERCISES

1. What is the greatest three-digit integer that has a factor of 19?

2. Let a "*prd*" number be defined as one in which the product of the positive divisors of the number, not including the number itself, is greater than the number. Which of the following is NOT a *prd* number?

(A) 12
(B) 18
(C) 27
(D) 45
(E) 20

3. If one of the positive factors of 120 is to be chosen at random, what is the probability that the chosen factor will not be a multiple of 5?

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) $\frac{1}{5}$

(E) $\frac{1}{6}$

$$x^y = 8192$$

4. In the equation above, x and y are positive integers. What is the smallest possible value of $x - y$?

5. How many positive integers less than 100 have an odd number of factors?

(A) 90
(B) 50
(C) 10
(D) 9
(E) 5

6. What is the sum of all of the prime factors of 71,400?

(A) 32
(B) 34
(C) 24
(D) 27
(E) 29

7. How many odd positive integers are factors of 480?

(A) 24
(B) 12
(C) 8
(D) 6
(E) 4

8. How many positive factors of 36 are also multiples of 4?

(A) 5
(B) 1
(C) 3
(D) 2
(E) 4

9. What is the smallest positive integer by which 120 can be multiplied so that the product will be a perfect square?

(A) 120
(B) 30
(C) 50
(D) 20
(E) 40

10. Find the largest prime factor of 493.

11. Find the number of even factors of 14014.

(A) 18
(B) 12
(C) 6
(D) 4
(E) 2

12. What is the largest prime factor of 1463?

13. How many positive integral factors does $2^8 \cdot 3^4 \cdot 7^6 \cdot 11$ have?

(A) 192
(B) 630
(C) 63
(D) 540
(E) 504

14. How many positive divisors does the number $2 \times 6^2 \times 5^3$ have?

(A) 48
(B) 24
(C) 15
(D) 12
(E) 6

15. If 2^k is a divisor of 2304, what is the largest possible value for k?

(A) 12
(B) 11
(C) 8
(D) 6
(E) 4

16. If 2^N is a factor of 20!, what is the largest possible value of N?

17. The number 596,505 can be expressed as a product $n \cdot m \cdot p$, where each of n, m, and p are two-digit numbers. Find $n + m + p$.

(A) 255
(B) 91
(C) 164
(D) 70
(E) 95

18. How many odd perfect square factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?

(A) 24
(B) 60
(C) 120
(D) 385
(E) 3850

19. How many perfect cube factors does $2^4 \times 3^6 \times 5^{10}$ have?

(A) 20
(B) 24
(C) 48
(D) 12
(E) 6

20. A rectangular quilt has 42 squares. How many shapes are there in which the quilt can be arranged?

21. What is the smallest positive integer by which 80 can be multiplied so that the product will be a perfect cube?

22. For how many positive integers n will $\frac{60}{n}$ also be an integer?

(A) 16
(B) 12
(C) 8
(D) 4
(E) 3

23. How many positive integer values of x are there such that $\frac{36}{x+3}$ is an integer?

(A) 9
(B) 8
(C) 6
(D) 4
(E) 3

5. SOLUTIONS TO WARM UP PROBLEMS

1. Solution: 12.
The number of positive factors is $(2 + 1)(1 + 1)(1 + 1) = 12$.

2. Solution: D.
$80 = 2^4 \times 5$ which has $(4 + 1)(1 + 1) = 10$ factors.
We also know that $80 = 2^4 \times 5 = 10(2^3)$ has $(3 + 1) = 4$ factors each is a multiple of 10.
So the probability is 4/10 = 2/5.

3. Solution:
We know that $9 \times 10 \times 11 \times 12 = 11800$ and $8 \times 9 \times 10 \times 11 = 7920$. We also know that $1 \times 2 \times 3 \times 4 = 24$.
We see that the first number in the product changes from 1 to 8. So there are 8 of them that are quadruple i-factorable.

4. Solution: 8191.
$x^y = 8192 = 2^{13} = (2^{13})^1$.
So $x = 2^{13} = 8192$ and $y = 1$. $x - y = 8192 - 1 = 8191$.

5. Solution: (C). 8.
Theorem: The number of factors of a prime number is 2. There are 8 prime number from 1 to 20 (2, 3, 5, 7, 11, 13, 17, 19). So the answer is 8.

6. Solution: (A). 10.
Any square number will have an odd number of factors. There are $\sqrt{100} = 10$ square numbers from 1 to 100. So the answer is 10.

7. Solution: 16.
$6006 = 2 \times 3 \times 1001 = 2 \times (3 \times 7 \times 11 \times 13)$.
The number of factors equals $(1 + 1)(1 + 1)(1 + 1)(1 + 1) = 16$.

8. Solution: 24.
$N = 6^2 \times 15 = (2 \times 3)^2 \times 3 \times 5 = 2^2 \times 3^3 \times 5$
Method 1: Any factor of N is the product of one or more primes 2, 3, or 5. We consider 2 first: we can use one 2, two 2's, and no 2's in the product. We have three choices.
Similarly we have four choices for 3 and two choices for 5.
We have $3 \times 4 \times 2 = 24$ combinations.
Method 2: The number of factors equals $(2 + 1)(3 + 1)(1 + 1) = 24$.

9. Solution: (B). 6.
$n = 2^3 \times 3^2 \times 5 = 2^3 \times (3^2 \times 5)$. We only need to calculate the number of factors for $(3^2 \times 5)$, which is $(2 + 1)(1 + 1) = 6$. So the number of even factors is 6.

10. Solution: 294.
Let m^3 be the perfect cube and n be the smallest positive integer.
$252 = 2^2 \times 7\times 3^2$
$252 \times n = 2^2 \times 7\times 3^2 \times n = m^3$.
$n = 2 \times 3 \times 7^2 = 294$.

11. Solution: (E). 12 (factors).
$21600 = 216\times 100 = (2 \times 3)^3 \times 10^2 = 2^3 \times 3^3 \times 2^2 \times 5^2 = 2^4 \times 3^2 \times 5^2 \times 2 \times 3 = (2^2 \times 3 \times 5)^2 \times 2 \times 3$.
We only need to calculate the number of factors for $(2^2 \times 3 \times 5)$, which is $(2 + 1)(1 + 1)(1 + 1) = 12$. So the number of perfect squares factors is 12.

12. Solution: (A). 24.
$2^4 \times 3^6 \times 5^{10} = 2^3 \times(3^2)^3 \times (5^3)^3 \times 2 \times 5 = (2^1 \times 3^2 \times 5^3)^3 \times 2 \times 5$.
We only need to calculate the number of factors for $(2^1 \times 3^2 \times 5^3)$, which is $(1 + 1)(2 + 1)(3 + 1) = 24$. So the number of perfect cube factors is 24.

13. Solution. (D). 8 (numbers)
$36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2$ has $3 \times 3 = 9$ factors. $n - 1$ must be one of the 9 factors in order for $\frac{36}{n-1}$ to be a natural number. Note that since n is positive integer, so $n - 1 > 0$, or $n > 1$. Among these 9 factors, the factor 1 should not be counted. So the answer is $9 – 1 = 8$.

6. SOLUTIONS TO EXERCISES

1. Solution: 988.
We know that $1000/19 \approx 52$. So $52 \times 19 = 988$. 988 is the greatest three-digit integer that has a factor of 19.

2. C

3. Solution: A.
$120 = 2^3 \times 3 \times 5$ which has $(3 + 1)(1 + 1) (1 + 1) = 16$ factors.
We also know that $120 = 5 \times (2^3 \times 3)$ has $(3 + 1) (1 + 1) = 8$ factors each is a multiple of 5.
So the probability is $8/16 = 1/2$.

4 Solution: – 11.
$x^y = 8192 = 2^{13} = (2)^{13}$.
So $x = 2$ and $y = 13$. $x – y = 2 – 13 = – 11$.

5. Solution: (D). 9.
All the square numbers have an odd number of factors. There are 9 square numbers from 1 to 99. The answer is then 9.

6. Solution: (B). 34.
$71400 = 100 \times 7 \times 102 = 100 \times 7 \times 3 \times 34 = 100 \times 7 \times 3 \times 34 = (2 \times 5)^2 \times 7 \times 3 \times 2 \times 17$.
The prime factors are 2, 3, 5, 7, and 17 and their sum is 34.

7. Solution: (E). 4.
$480 = 2^5 \times 3 \times 5 = 2^5 \times (3^1 \times 5^1)$
The number of odd factors equals $(1 + 1)(1 + 1) = 4$.

8. Solution: (C). 3.
Since $36 = 2^2 \times (3^2)$, the number of positive factors of 36 that are also multiples of 4 is $(2 + 1) = 3$.

9. Solution: (B). 30.
Let m^2 be the perfect cube and n be the smallest positive integer.
$120 = 12 \times 10 = 3 \times 4 \times 2 \times 5 = 2^3 \times 3 \times 5$.
$80 \times n = 2^3 \times 3 \times 5 \times n = m^2$.
$n = 2 \times 3 \times 5 = 30$.

10. Solution: 29.
Since $22^2 = 484 < 4939$ and $23^2 = 529 > 4939$, the square root of 493 is between 22 and 23.
So we use the prime number 19, 17, 13, 11, 7, and 3 to test and we get:
$493 = 17 \times 29$.

11. Solution: (B). 12.
$14014 = 2 \times 7007 = 2 \times 7 \times 1001 = 2 \times 7 \times 7 \times 11 \times 13 = 2 \times (7^2 \times 11 \times 13)$.
There are $(2 + 1)(1 + 1)(1 + 1) = 12$ even factors.

12. Solution: 19.
1463 is divisible by both 7 and 11 so it is divisible by 77.
$1463 = 77 \times 19 = 11 \times 7 \times 19$

13 Solution: (B). 630.
The number of divisors $d = (8 + 1)(4 + 1)(6 + 1)(1 + 1) = 630$.

14. Solution: (A). 48.
Note that $2 \times 6^2 \times 5^3 = 2 \times 2^2 \times 3^2 \times 5^3 = 2^3 \times 3^2 \times 5^3$.
The number of divisors $d = (3 + 1)(2 + 1)(3 + 1) = 48$.

15. Solution: (C). 8.
$2304 = 16 \times 144 = 2^4 \times (12)^2 = 2^4 \times (3 \times 2^2)^2 = 2^4 \times 3^2 \times 2^4 = 2^8 \times 3^2$. $k = 8$.

16. Solution: 18.
$$N = \left\lfloor \frac{20}{2} \right\rfloor + \left\lfloor \frac{20}{2^2} \right\rfloor + \left\lfloor \frac{20}{2^3} \right\rfloor + \left\lfloor \frac{20}{2^4} \right\rfloor = 10 + 5 + 2 + 1 = 18.$$

17. Solution: (A). 255.
$596{,}505 = 5 \times 119301 = 5 \times 3 \times 39767 = 5 \times 3 \times 7 \times 5681 = 5 \times 3 \times 7 \times 13 \times 437 = 5 \times 3 \times 7 \times 13 \times 19 \times 23 = (3 \times 23) \times (5 \times 19) \times (7 \times 13) = 69 \times 95 \times 91$
The sum is $69 + 95 + 91 = 255$.

18. Solution: (C). 120.
$$2^4 \times 3^6 \times 5^{10} \times 7^9 \quad \Rightarrow \quad 3^6 \times 5^{10} \times 7^9 \quad \Rightarrow \quad (3^3 \times 5^5 \times 7^4)^2 \times 7$$
The number of odd perfect square divisors is the same as the number of divisors for $3^3 \times 5^5 \times 7^4$ which can be calculated as $(3 + 1)(5 + 1)(4 + 1) = 120$.

19. Solution: (B). 24.
$2^4 \times 3^6 \times 5^{10} = (2 \times 3^2 \times 5^3)^3 \times 2 \times 5$
The number of factors is $(1 + 1)(2 + 1)(3 + 1) = 24$.

20. Solution: 4 ways.
This is the same way of asking how many ways can 42 be expressed as a multiple of two numbers. $42 = 1 \times 42 = 2 \times 21 = 3 \times 14 = 6 \times 7$.

21. Solution: 100.
$80 = 2^4 \times 5$.
The smallest positive integer is $2^2 \times 5^2 = 100$.

22. Solution: (B). 12.
$60 = 6 \times 10 = 2^2 \times 3 \times 5$ has 12 factors. n must be one of them in order for $\frac{60}{n}$ to be an integer. So the answer is 12.

23. Solution: (C). 6.
$36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2$ has $3 \times 3 = 9$ factors. $x + 3$ must be one of the 9 factors in order for $\frac{36}{x+3}$ to be an integer. Note that since x is positive integer, so $x + 3 \geq 4$. Among these 9 factors, three of them are less than 4. So the answer is $9 - 3 = 6$.