BASIC KNOWLEDGE for Magic Squares

A magic square is an n by n square of order n with an arrangement of n^2 numbers.

These numbers are arranged so that the sum of each row, each column, and each diagonal of the square is the same. This sum is known as the magic sum, S.

Example of a magic square:

8	3	4
1	5	9
6	7	2

In the magic square above, the magic sum is 15.

(1) Magic Sum Formula

If an order n magic square begins with the entry m, and its entries are consecutive counting numbers, its magic sum can be calculated by the formula

$$S = \frac{n(2m + n^2 - 1)}{2}$$

If a magic square of order n has entries 1, 2, 3,4…..n^2, then the magic sum S can be calculated by plugging in m=1 into the above formula to obtain:

$$S = \frac{n(n^2 + 1)}{2}$$

For example, for a 3 by 3 magic square where n=3, the magic sum is $S = \frac{3(3^2 + 1)}{2} = 15$.

(2) For a magic square, with the magic sum S, where a, b, and c are placed as shown in the figure below,

$$c = \frac{S}{3} = \frac{a + b}{2}$$

	c	

a	c	b

a		
	c	
		b

(3) For a magic square, where a, b, and c are placed as shown in the figure below,

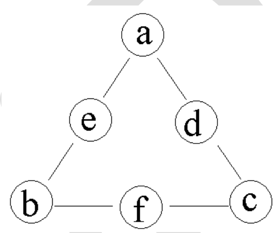
$$c = \frac{a+b}{2}$$

c		
		a
	b	

(4). A magic triangle is a triangle like the one pictured below where the sum of the numbers in each of the 3 sides is the same.

Let k be the sum of the numbers in the vertices and S be the magic sum. We have the following relationship:

$$(a+b+c+d+e+f)+k=3S$$



PROBLEMS

1. In the magic square, the sum of the three numbers in any row, column or diagonal is the same. Find the sum of the three numbers in any row. In other words, find the magic sum.

$2x$	3	2
		-3
0	x	

2. Find the value of x in the magic square:

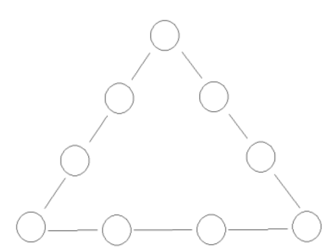
34		
	x	
		56

3. The remaining six boxes in the square below are filled so that the sum of the numbers in each row, column, and diagonal is the same. What number goes in the upper left hand corner?

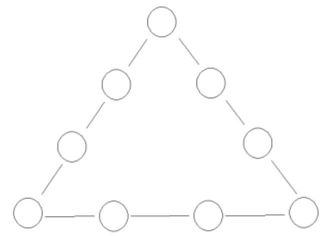
		6
		7
4		

4. The whole numbers 11 through 19 are arranged in the squares shown so that the sum of the numbers in each row, column and diagonal is the same. What is the common sum? (1997 Mathcounts State Sprint)

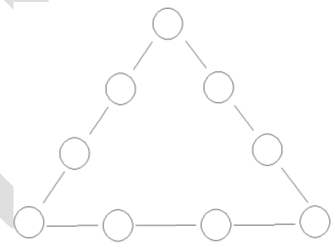
5. The numbers 1, 2, 3… ,9 are arranged, one per circle, in the triangle shown so that the sum, S, of the numbers on each side of the triangle is the same. The sum of the numbers in the three corner circles is 12. What is the value of S? (1997 Mathcounts Chapter Sprint #27)



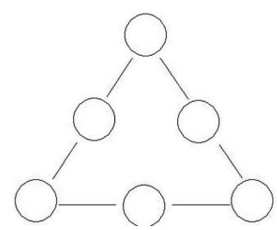
6. The numbers 1, 2, 3… 9 are arranged, one per circle, in the triangle shown so that the sum of the numbers in the four circles along each side of the triangle is 17. What is the sum of the numbers in the three circles at the vertices of the triangle? (Mathcounts)



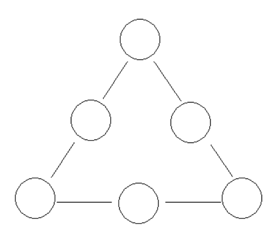
7. The digits 1, 2, 3, …, 9 are arranged, one per circle, in the triangle shown so that the sum of the numbers on each side of the triangle are the same. What is the difference between the greatest and the least sums possible? (1997 Mathcounts Nationals Sprint #26)



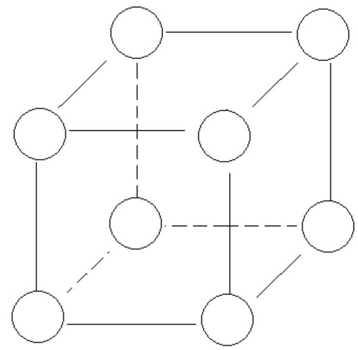
8. The numbers 10, 11, 12, …, 15 are arranged, one number per circle, in the triangle shown so that the sum of the numbers on each side of the triangle are the same. What is the greatest sum possible?



9. The digits 1, 2, 3, …, 6 are arranged, one per circle, in the triangle shown so that the sum of the numbers on each side of the triangle are the same. How many different magic triangles can be made?



10. Eight of the digits 1, 2, 3, …, 9 are arranged, one per circle with each circle on one of the eight edges of the cube, in the cube shown so that S, the sum of the numbers on each face of the cube is the same. Seven of the eight numbers are labeled, but one. It is known that S is not divisible by the missing number. What is the missing number x?



11. In a magic square, the sum of the entries in any row, column, or diagonal is the same. The figure shows four of the entries of a magic square. Find x. (1996 AIME)

X	**19**	**96**
1		

12. Prove: For a magic square, with the magic sum S, where a, b, and c are placed as shown in the figure below, $c = \frac{S}{3} = \frac{a+b}{2}$

a		
	c	
		b

13. Prove: For a magic square, where a, b, and c are placed as shown in the figure below, $c = \frac{a+b}{2}$

c		
		a
	b	

SOLUTIONS

Problem 1. Solution: **3**

$\frac{-3+x}{2}=2x \quad \Rightarrow \quad x = -1$

-2 + 3 + 2 = 3.

Problem 2. Solution: **45**

Since $x=\frac{S}{3}$

$3x=34+x+56$, $x=45$

Problem 3. Solution: **8**

We calculate *a* first. $a=\frac{S}{3}=\frac{4+6+a}{3}$. $a=5$.

So S = 15 and b = 15 - 7- 6 = 2 and c = 15 – 5 – 2 = 8

c		**6**
	a	**7**
4		**b**

Problem 4. Solution: **45**

$$S=\frac{n(2m+n^2-1)}{2}=\frac{3(2\times 11+9-1)}{2}=45$$

Problem 5. Solution: **19**

Let the sum of the numbers in the vertices be k and the magic sum be S.

$(1+2+3+4+5+6+7+8+9)+k=3S$, $45+k=3S$

Since k = 12, we plug in k in the above equation to obtain $S=\frac{45+12}{3}=19$

Problem 6. Solution: **6**

This problem is very similar to the previous problem and so we solve it using the same way.

Let the sum of the numbers in the vertices be k and the magic sum be S.

$(1+2+3+4+5+6+7+8+9)+k=3S$, $45+k=3S$
Since S = 17, we plug in S to the above equation and get $k=3\times 17-45=6$

Problem 7. Solution: **6**

Let the sum of each row be S and let a, b, and c be the three numbers located in the corners.
We have $1+2+3+4+5+6+7+8+9+k=3S$ or $S=\frac{45+k}{3}$
The greatest value for k is when a, b, and c are maximized, so: 9+8+7 =24 is the greatest k and the smallest k is 3+2+1 =6
The greatest sum of $S=\frac{45+24}{3}=23$ and
the least sum of $S=\frac{45+6}{3}=17$
Since the question asks for the difference between the greatest and the least sums, so the answer is
23 – 17 = 6

Problem 8. Solution: **39**

Let the sum of each row be S and let a, b, and c be the three numbers located in the corners.
We have $10+11+12+13+14+15+k=3S$ or $S=\frac{75+k}{3}$
The greatest value for k is when a, b, and c are maximized, so: max k = 15+14+13 =42

The greatest sum is then $S=\frac{75+42}{3}=39$

Problem 9. Solution: **4 different magic squares**

Let the sum of the numbers in the vertices is k and the magic sum is S.

$(a+b+c+d+e+f)+k=3S$

Or

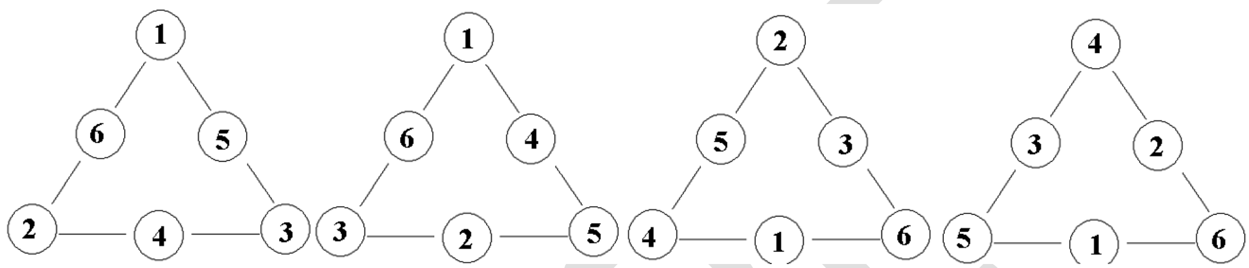
$(1+2+3+4+5+6)+k=3S$

$21 + k = 3S$

Since k is the sum of three numbers, so the smallest value for s is 1 +2 + 3 = 6 and the greatest value for s is 4+5+6 = 15.

Therefore, we get $9 \le S \le 12$ and 4 groups of s and k:

$\begin{cases} S = 9 \\ k = 6 \end{cases}$ $\begin{cases} S = 10 \\ k = 9 \end{cases}$ $\begin{cases} S = 11 \\ k = 12 \end{cases}$ $\begin{cases} S = 12 \\ k = 15 \end{cases}$



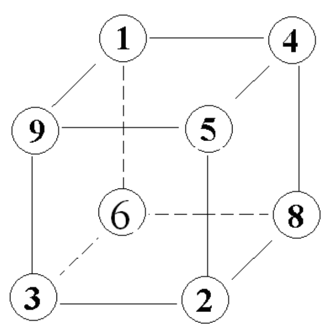
Problem 10. Solution: **7**

1+2+3+4+5+6+7+8+9-x= 45-x

Since every vertex belongs three faces, so the sum of numbers in six faces are 6S = 3(45-x).

Or 2S = 45 - x.

We are sure that x is odd and since S is not divisible by x, we concluded that x = 7, and S = 19.



Problem 11. Solution: **200**

Step 1.

Using the formulas from our chapter discussion, we can obtain the lower right hand corner number.

The lower right hand corner = $\frac{19+1}{2}=10$

X	**19**	**96**
1		
		10

Step 2.

Using the same formula in step 1, we can find the value of the square to the left of the square we just found:

$\frac{\mathbf{191}+1}{2}=96$

X	**19**	**96**
1		
	191	**10**

Step 3.

Using the most basic definition of a magic square, we can write

X + 1 + ? = ? + 191 + 10

Solve for X to get:

X = 200.

12. Solution: We give a short proof as follows:

m	S-m-n	n
	c	
S-c-n	x	S-c-m

According to the definition of a magic square, we have:

$$S - c - n + x + S - c - m = x + c + S - m - n \quad \Rightarrow \quad c = \frac{S}{3}$$

and:

$S - c - n + n = S - c = 3c - c = 2c$ Let a = n, and b = S-c-n, we have $c = \frac{a+b}{2}$.

13. Solution: A short proof is as follows:

c	$\frac{2S}{3} - a$	x
	$\frac{S}{3}$	b
	a	$\frac{2S}{3} - c$

According to the definition of a magic square, we have:

$$c + \frac{2S}{3} - a + x = x + b + \frac{2S}{3} - c \quad \Rightarrow \quad c = \frac{a+b}{2}$$