BASIC KNOWLEDGE

Statements

A statement is any sentence that is either true or false, but not both.

Examples:

Boston is a city in USA.
$1 + 1 = 3$
A spider does not have six legs.

The following sentences are not statements:

Do your homework.	(a command)
How do you solve this math problem?	(a question)
Mathcounts contest is harder than the AMC 8 contest.	(an opinion)
This sentence is false.	(a paradox)

Negations

The sentence "AMC 8 contest consists of 25 problems" is a statement; the negation of this statement is "AMC8 contest does not consist of 25 problems".
The negation of a true statement is false, and the negation of a false statement is true.

Statement	Negation
All do	Some do not (Not all do)
Some do	None do (All do not)

Examples: Form the negation of each statement:

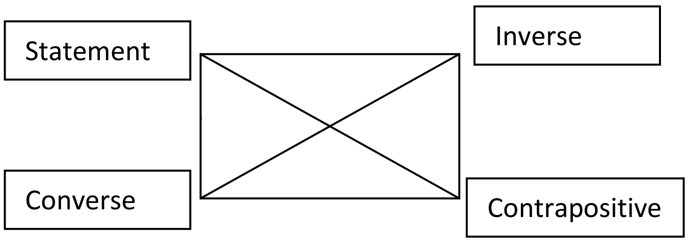
The moon is not a star.	$\Rightarrow$	The moon is a star.
The moon is a star.	$\Rightarrow$	The moon is not a star.
A spider does not have six legs.	$\Rightarrow$	A spider has six legs.
Some rabbits have short tails.	$\Rightarrow$	No rabbit has a short tail.
Some rabbits do not have short tails.	$\Rightarrow$	All rabbits have short tails.
No rabbit has a short tail.	$\Rightarrow$	Some rabbits have short tails.

Converse, Inverse, and Contrapositive

Direct statement	If p, then q.
Converse	If q, then p.
Inverse	If not p, then not q.
Contrapositive	If not q, then not p.

Direct statement	If I live in Boston, then I live in USA.
Converse	If I live in USA, then I live in Boston.
Inverse	If I do not live in Boston, then I do not live in USA.
Contrapositive	If I do not live in USA, then I do not live in Boston.

Rectangle of logical equivalent



Logically equivalent pair of statements (diagonally opposite):

A statement and its contrapositive
The inverse and converse of the same statement

Not logically equivalent pair of statements (adjacent):

A statement and its inverse
A statement and its converse
The converse and contrapositive of the same statement
The inverse and contrapositive of the same statement

Examples:

Statement:	A square is a rectangle	(true)
Converse	A rectangle is a square	(false)
Inverse	A figure that is not a square is not a rectangle	(false)
Contrapositive	A figure that is not a rectangle s is not a square	(true)

Euler Diagram

Deductive reasoning consists of three steps as follows:

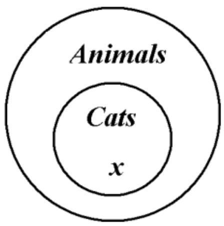
(1). Making a general statement (major premise).
(2). Making a particular statement (minor premise).
(3). Making a deduction (conclusion).

Example:

(1). The major premise is: All cats are animals
(2). The minor premise is: Jerry is a cat.
(3). The conclusion is: Jerry is an animal.

Procedures to draw the diagram:

(1) Draw a big circle to represent the first premise. This is the region for "animals".
(2) Draw a second circle to represent "all cats". Since all cats are animals, the second circle goes inside the first big circle.
(3) Put Jerry inside where it belongs. The second premise stated that Jerry is a cat. Put Jerry inside the region marked "Cats".

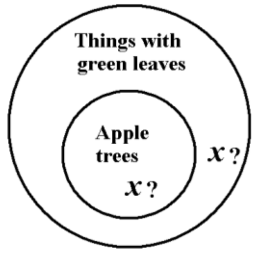


x represents Jerry

Example: Is the following argument valid? An argument is valid if that the premises are true and these premises force the conclusion to be true.

All apple trees have green leaves
That plant has green leaves.
That plant is an apple tree.

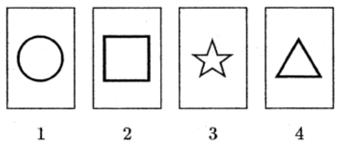
Solution: we draw the Euler Diagram. We see that "that plant" can go either inside the small circle or outside it. So the argument is not valid.



Some Problem Solving Skills

(a) Find the contrapositive of the statement.

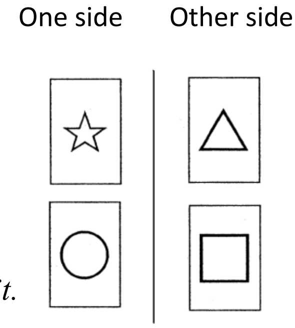
Example: Each card has either a circle or a star on one side and either a triangle or a square on the other side. In order to verify the statement "every card with a star on it also has a triangle on it," which numbered card(s) must be turned over? (Mathcounts)



Solution: two cards (cards 2 and 3).
We introduce in this section a two-step method. This method can be used to solve any similar problems.

Step 1. We verify the statement first:

Every card with a star on it also has a triangle on it.



We must turn over every card with a star on it (card 3) to make sure it has a triangle on the other side.

Step 2. We then verify the contrapositive of the statement:

Every card without a triangle on it also does not have a star on it.

We must turn over any card without a triangle on it (in this case, card 2 with a square as shown in the figure on the left) to make sure it doesn't have a star on the other side).

(b) Find two statements that are contradicted to each other

Example: There are three boxes with different colors: red, yellow and blue. One apple is in one of the three boxes.
Only one of the following statements is true, and the others are false.

I: Apple is in the red box; II Apple is not in the yellow box, and III: Apple is not in the red box.

Which box is the apple in?

Solution: First we find the two statements that are contradicted to each other. There must be a true statement between these two. Other statements left are all false.

Statement I and Statement III are two contradicted statements. We are sure that the true statement is one of these two statements, although we do know which one. So we conclude that the statement II is false. Then we know the apple is in the yellow box.

(c) Focus on the step before the last.

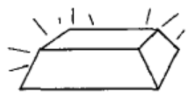
Example: A turtle crawls up a 12 foot hill after a heavy rainstorm. The turtle crawls 4 feet, but when it stops to rest, it slides back 3 feet. How many tries does the turtle make before it makes it up the hill?

Solution: 9.
We look at where the turtle was just before the last try. Since the turtle can crawl 4 feet each time, $12 - 4 = 8$. Every try the turtle goes up 1 foot. It takes the turtle 8 tries when it reaches the 8 feet location. The turtle needs one more try to reach the top. Note when it reaches the top, there is no sliding back.

(d). Dividing into three groups.

When you need to weigh a number of coins with counterfeit coin, divide the coins into three groups with the number of coins in each group: m, m, m, or $m, m, m - 1$ or $m, m, m + 1$.

Example: A jeweler has four small bars that are supposed to be gold. He knows that one is counterfeit and the other three are genuine. The counterfeit bar has a slightly different weight than a real gold bar. Using a balance scale, what is the minimum number of weighings necessary to guarantee that the counterfeit bar will be detected? (Mathcounts)



Solution: 2.

We divide the four bars into three groups: 1, 1, and 2. We weight two bars, say, bar A and bar B, first.

Case I: If their weights are different, we remove one, say, bar A, and put a third bar, say bar C. If B and C are the same, and then bar A is the counterfeit. If bar B and bar C are different, bar B is the counterfeit (since it's weight is different from both A and C).

Case II: If their weights are the same, then we remove one, say, Bar A, and put a third bar, say Bar C. If B and C are the same, then Bar D is the counterfeit. If Bar B and Bar C are different, Bar C is the counterfeit.

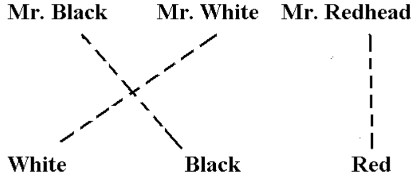
So two weighings are necessary.

(e). Drawing Solid and Dash Lines

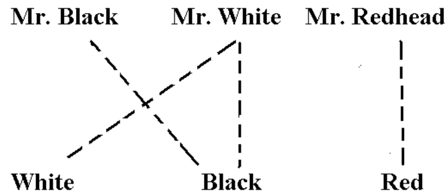
Example: Three friends – math teacher Mr. White, science teacher Mr. Black, and history teacher Mr. Redhead – met in a cafeteria. "It is interesting that one of us has white hair, another one has black hair, and the third has red hair, though no one's name gives the color of their hair" said the black-haired person. "You are right," answered White. What color is the history teacher's hair?

Solution: The history teacher's hair is black.
If the relationship of two things is certain, we draw a solid line between them. Otherwise, we draw a dash line.

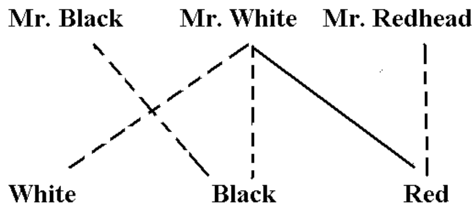
We know that no one's name gives the color of their hair. So we draw the dash lines as shown on the right:



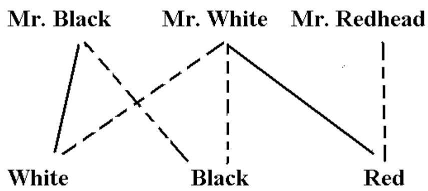
We know that Mr. White answered the black-haired person. So he has no black hair. We draw a dash line between Mr. White and "black hair".



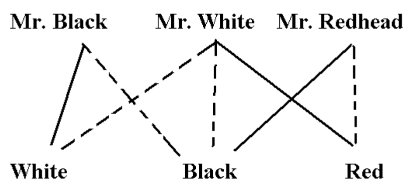
So Mr. White must have red hair. We draw a solid line to indicate that Mr. White has red hair.



Mr. Black cannot have black hair, so he must have white hair. We draw a solid line for that.



We know for sure that the history teacher's hair is black.



(f). Making a chart

Example: Each of three marbles A, B, and C, is colored one of the three colors. One of the marbles is colored white, one is colored red, and one is colored blue. Exactly one of these statements is true:

1) A is red. 2) B is not blue. 3) C is not red.
What color is marble B? (Mathcounts Competitions).

Solution:

Case I: 1) is true.

Since exactly one of the three statements is true, so both 2) and 3) must be false. Therefore B is blue and C is red. We know that only one marble is colored red, so 1) is not true.

		Red	White	Blue
True	*A*	√		
False	*B*			√
False	*C*	√		

Case II: 2) is true and both 1) and 3) are false.

We know that B can be red or white.

If B is red, C is also red. Contradiction!

		Red	White	Blue
False	*A*			
True	*B*	√		
False	*C*	√		

If B is white, C is red and A is blue. Works!

		Red	White	Blue
False	*A*			√
True	*B*		√	
False	*C*	√		

Case II: 3) is true and both 1) and 2) are false.

We know that *C* can be blue or white.

If *C* is white, B is blue. *A* is not red. So A must be white or blue. Contradiction!

		Red	White	Blue
False	*A*		?	?
False	*B*			√
True	*C*		√	

If *C* is blue, then *B* is blue. Contradiction!

		Red	White	Blue
False	*A*			
False	*B*			√
True	*C*			√

The answer is that *B* is white.