BASIC KNOWLEDGE

(1). RATIOS:

Ratios are used to compare two or more numbers.

For any two numbers *a* and *b* ($b \neq 0$), the ratio is written as $a : b = a \div b = \frac{a}{b} = a / b$.

Example 1: If 24 students in a class of 30 students were present, what percent of the students were absent? (Mathcounts Handbooks).

Solution: 20%.

The number of student who were absent: 30 – 24 = 6.

6/ 30 = 0.2 = 20%.

Example 2: There are 16 girls in a class of 30 students. Find the ratio of girls to boys. Express your answer as a common fraction. (Mathcounts Handbooks).

Solution: 8/7.

The number of boys in the class is 30 – 16 = 14. $\Rightarrow$ 16/14 = 8/7.

Properties of ratios:

The first term of a ratio can be any number. The second term can also be any number except zero.

If the two terms are multiplied by the same number *d*, the ratio does not change.

$a : b = (a \times d) : (b \times d)$

Example: $3 : 7 = (3 \times 5) : (7 \times 5) = 15 : 35$

If the two terms are divided by the same number *c* ($c \neq 0$), the ratio does not change.

$a : b = (a \div c) : (b \div c)$

Example 1: $10 : 15 = (10 \div 5) : (15 \div 5) = 2 : 3$

Example 2: $\frac{5}{6}:\frac{10}{13}=(1\times 6):(2\times 13)=13:12$

If the total number of parts is $m = A + B$, and $A : B = a : b$, then

the fractional part of a is $\frac{a}{a+b}$, and

the fractional part of b is $\frac{b}{a+b}$.

$$A=\frac{a}{a+b}\times m, \text{ and } B=\frac{b}{a+b}\times m.$$

If the total number of parts is $m = A + B + C$, and $A : B : C = a : b : c$, then

the fractional part of a is $\frac{a}{a+b+c}$,

the fractional part of b is $\frac{b}{a+b+c}$, and

the fractional part of c is $\frac{c}{a+b+c}$.

$$A=\frac{a}{a+b+c}\times m,\ B=\frac{b}{a+b+c}\times m, \text{ and } C=\frac{c}{a+b+c}\times m.$$

Example 1: A certain paint color is created by mixing 3 parts of red with every 5 parts of blue. How many gallons of red paint are needed to mix 40 gallons of this color? (Mathcounts Handbooks).

Solution: 15

$$A=\frac{a}{a+b}\times m=\frac{3}{3+5}\times 40=15.$$

Example 2: In a group of 72 students if the ratio of boys to girls is 5: 3, how many boys are in the group?

Solution: 45.

$$A = \frac{a}{a+b} \times m = \frac{5}{5+3} \times 72 = 45.$$

Example 3: Keith bought paper for making origami figure. He bought 2 packages of orange paper, 3 packages of yellow paper, and 5 packages of blue paper. What fraction of the papers was blue?

Solution: $\frac{1}{2}$.

$$\frac{a}{a+b+c} = \frac{5}{2+3+5} = \frac{5}{10} = \frac{1}{2}$$

Example 4: Keith bought 10 packages of paper for making origami figure. The ratio of orange paper, yellow paper, and blue paper is 2 : 3 : 5. How many packages of blue paper did he buy?

Solution: 5.

$$A = \frac{a}{a+b+c} \times m = \frac{5}{2+3+5} \times 10 = 5$$

(2) RATES:

A rate is a ratio used to compare two numbers of different units. If the second term of the ratio is 1, the rate is called a unit rate.

Example 1: Sam drove 100 miles in 2 hours. What are his rate and the unit rate?

Solution: The rate is 100 miles/2 hours and the unit rate is 50/1 or 50 miles per hour.

Example 2: Michael types 250 words in 20 minutes. How many hours will it take him to type a 7500 word paper? (Mathcounts Handbooks).

Solution: 10.

The unit rate is 250 ÷ 20 = 12.5 words per minute.

The time to type 7500 words is 7500 ÷ 12.5 = 600 minutes = 10 hours.

Example 3: A car gets 27 miles per gallon. How many miles will it go on 9 gallons of gas? (Mathcounts Handbooks).

Solution: 243.

The number of miles will the car go is $27 \times 9 = 243$.

Example 4: A basketball player makes 80% of the shots he attempts in each game. In a certain game, he made 20 of his shots. How many shots did he attempt in the game? (Mathcounts Handbooks).

Solution: 25.

Let x be the total number of shots he made.

$0.8 \times x = 20. \qquad \Rightarrow \qquad x = 25.$

Example 5: A pork roast should be cooked 50 minutes per pound. How many hours should a 6-pound roast be cooked? (Mathcounts Handbooks).

Solution: 5.

The number of hours it takes is $50 \times 6 = 300$ minutes = 6 hours.

(3). PROPORTIONS:

A proportion is an equation of two ratios. For example, $\frac{a}{b} = \frac{c}{d}$. We can find a if we know b, c, and d or we know b and the value of c/d.

Properties of Of Proportion:

Property 1: $\frac{a}{b} = \frac{c}{d}$ is equivalent to :

$$ad = bc, \qquad \frac{a}{c} = \frac{b}{d}, \qquad \frac{d}{b} = \frac{c}{a}, \qquad \frac{b}{a} = \frac{d}{c}.$$

Example 1: $\frac{8}{6}=\frac{4}{3}$ $\Rightarrow$ $8\times 3=4\times 6$ $\frac{8}{4}=\frac{6}{3}$ $\frac{3}{6}=\frac{4}{8}$ $\frac{6}{8}=\frac{3}{4}$

Example 2: The ratio of width to length of a rectangular room is $\frac{4}{3}$ and the width is $8\frac{7}{10}$. What is the length?

Solution: $\frac{L}{W}=\frac{4}{3}$ $\Rightarrow$ $L=\frac{4}{3}W=\frac{4}{3}\times 8\frac{7}{10}=\frac{4}{3}\times\frac{87}{10}=\frac{58}{5}$

Property 2: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$

Example: $\frac{21}{7}=\frac{30}{10}$ $\Rightarrow$ $\frac{21+7}{7}=\frac{30+10}{10}$ (= 4) and $\frac{21-7}{7}=\frac{30-10}{10}$ (= 2).

Property 3: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Example: $\frac{21}{7}=\frac{30}{10}$ $\Rightarrow$ $\frac{21+7}{21-7}=\frac{30+10}{30-10}$ ($\frac{28}{14}=\frac{40}{20}$ =2).

Property 4: If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\ldots=\frac{a_n}{b_n}$

Then $\frac{a_1+a_2+a_3+\ldots+a_n}{b_1+b_2+b_3+\ldots+b_n}=\frac{a_1}{b_1}$

Example 1: Find x if $\frac{2x-y}{5}=\frac{x+y}{10}=\frac{3}{5}$.

Solution: 3.

$\frac{2x-y}{5}=\frac{x+y}{10}=\frac{(2x-y)+(x+y)}{15}=\frac{3x}{15}=\frac{3}{5}$ $\Rightarrow$ x = 3.

Example 2: If $\frac{y}{x-z}=\frac{x+y}{z}=\frac{x}{y}$ for three positive numbers *x*, *y*, and *z*, all different, then what is the value of $\frac{x}{y}$? (1992 AMC).

Solution: 2.

$$\frac{x}{y}=\frac{y}{x-z}=\frac{x+y}{z} \quad \Rightarrow \quad \frac{x}{y}=\frac{x+y+(x+y)}{y+(x-z)-z}=\frac{2x+2y}{x+y}=\frac{2(x+y)}{x+y}=2.$$

Proof of properties (2) and (3)

Since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a}{b}+1=\frac{c}{d}+1 \quad \Rightarrow \quad \frac{a}{a}+\frac{a}{b}=\frac{c}{d}+\frac{d}{d} \quad \Rightarrow$

$$\frac{a+b}{b}=\frac{c+d}{d} \qquad (1)$$

We also have $\frac{a}{b}-1=\frac{c}{d}-1 \quad \Rightarrow \quad \frac{a}{b}-\frac{a}{a}=\frac{c}{d}-\frac{d}{d} \quad \Rightarrow$

$$\frac{a-b}{b}=\frac{c-d}{d} \qquad (2)$$

(1) ÷ (2), $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (note $a \neq b$ and $c \neq d$). (3)

(4). CONTINUED RATIO

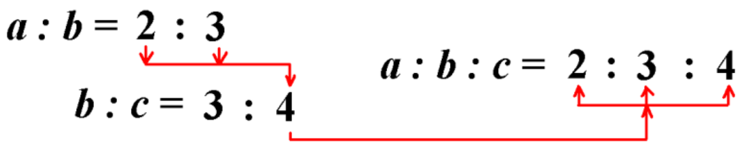
The ratio of three or more quantities is called the continued ratio. For example, *a:b:c* is a combinations of three separated ratios ⇒ *a:b*, *a:c*, and *b:c*.

(1) If *a : b : c* = 2 : 3 : 4, then

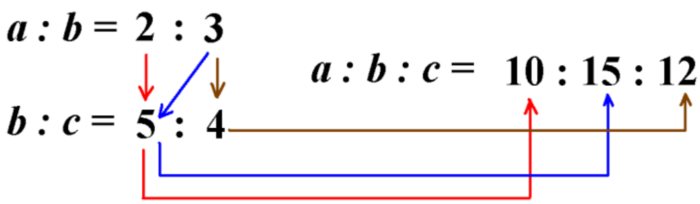
a : b = 2 : 3, *b : c* = 3 : 4, and *c : a* = 4 : 2.

(2) If *a : b* = 2 : 3, *b : c* = 3 : 4, and *c : a* = 4 : 2, then

a : b : c = 2 : 3 : 4,



(3) If $a : b = 2 : 3$, and $b : c = 5 : 4$ (note $3 \neq 5$), then $a : b : c = (2 \times 5) : (3 \times 5) : (3 \times 4) = 10 : 15 : 12$.



Example 1: Three numbers *a*, *b*, and *c* in the ratios of $a : b = 3 : 4$ and $b : c = 5 : 6$ have a sum of 118. What are the values of *a*, *b*, and *c*?

Solution: $a = 30$, $b = 40$, and $c = 48$.

Method 1:

$a:b=3:4=15:20$ and $b:c=5:6=20:24$. By the property 2 of the continued ratio, we get: $a:b:c=15:20:24$.

We also know that $a+b+c=118$, so $a=\dfrac{15}{15+20+24}\times 118=30$,

Similarly, $b = 40$, and $c = 48$.

Method 2:

$a:b=3:4$ and $b:c=5:6$

By the property 3 of the continued ratio, we get: $a : b : c = 15 : 20 : 24$.

so $a=\dfrac{15}{15+20+24}\times 118=30$, and $b = 40$, and $c = 48$.

Example 2: Machine A can fill 1 box of nails in 6 minutes. Machine B can fill 1 box of nails in 9 minutes. They started to work at the same time and they stopped also at the same time. Total they filled 100 boxes. How many were filled by machine A?

Solution: 60.

Method 1: Machine A would fill 3 boxes of nails in 18 minutes. Machine B would fill 2 boxes of nails in 18 minutes. So the ratio of their work is 3 : 2.

The number of boxes filled by machine A is:

$$\frac{3}{3+2}\times 100 = 60$$

Method 2: Since the ratio of their work is 3 : 2, let the number of boxes filled by machines A be $3x$, and the number of boxes filled by machines B be $2x$.

$3x + 2x = 100 \quad \Rightarrow \quad x = 20 \quad \Rightarrow \quad 3x = 60.$

Example 3: Alex paid \$945 to transport his animals by ferry. The costs are \$3, \$2 and \$1 for each cats, dog, and squirrel, respectively. The ratios of cats to dogs is 2 : 9, and dog to squirrel 3 : 7. How many cats were there?

Solution:

The ratio of the number of animals can be obtained as follows:

$c : d = 2 : 9$ and $d : s = 3 : 7 \quad \Rightarrow \quad c : d : s = 6 : 27 : 63 = 2 : 9 : 21.$

Then the ratio of the cost is then:

$(3 \times 2) : (2 \times 9) : (1 \times 27) = 2 : 6 : 7.$

So the cost for cats is calculated as follows:

$$\frac{2}{2+6+7}\times 945 = 126$$

The number of cats is $126 \div 3 = 42$.