

CAUCHY'S INEQUALITY

Problem 1. (Mathcounts State) If x , y , and z are positive integers such that $x^2 + y^2 + z^2 = 174$. What is the greatest value of $x + y + z$?

Problem 2. If $x + 2y = 1$, find the smallest value of $\frac{1}{x} + \frac{1}{y}$. $x, y \in \mathbb{R}^+$.

Problem 3. (AMC) Find the minimum value of $\sqrt{x^2 + y^2}$ if $5x + 12y = 60$.

Problem 4. (ARML) If a , b , and c are each positive numbers, and $a + b + c = 6$, show that $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq \frac{75}{4}$.

Problem 5. (ARML) If a , b , c , and d are each positive, $a + b + c + d = 8$, $a^2 + b^2 + c^2 + d^2 = 25$, and $c = d$, compute the greatest value that c can have.

Problem 6. (USAMO) The sum of 5 real numbers is 8 and the sum of their squares is 16. What is the largest possible value for one of the numbers?

Problem 7. (2018 Stanford Math Tournament) Let x , y , z be non-negative real numbers satisfying $xyz = \frac{2}{3}$. Compute the minimum value of $x^2 + 6xy + 18y^2 + 12yz + 4z^2$.

Problem 8. (1984 AMC Problem 29) Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy $(x - 3)^2 + (y - 3)^2 = 6$.

- (A) $3 + 2\sqrt{2}$ (B) $2 + \sqrt{3}$ (C) $3\sqrt{3}$ (D) 6 (E) $6 + 2\sqrt{3}$

Problem 9. (1996 AMC Problem 25) Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have?

- (A) 72 (B) 73 (C) 74 (D) 75 (E) 76

Problem 10. (2004 AMC12 B) The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy -plane. Let a and b be the maximum and minimum values of y/x over all points (x, y) on the ellipse. What is the value of $a + b$?

- (A) 3 (B) $\sqrt{10}$ (C) $\frac{7}{2}$ (D) $\frac{9}{2}$ (E) $2\sqrt{14}$

Problem 11. (IMO) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Keys:

1. Answer: 22.

2. Answer: $(1 + \sqrt{2})^2$.

3. Answer: $\frac{60}{13}$.

4. Answer:

5. Answer: $\frac{7}{2}$.

6. Answer: $\frac{16}{5}$.

7. Answer: 18

8. Answer: (A).

9. Answer: (B).

10. Answer: (C).

11. Answer: