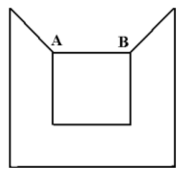
1999 Mathcounts National Sprint Round Solutions

1. Solution: $\frac{5}{12}$.

A 3-digit number is divisible by 3 if the sum of its digits is divisible by 3.
The first digit cannot be 0, so we have the following four groups of 3 such that the three different numbers sum to a multiple of 3:
2, 4, 0; 8, 4, 0; 6, 4, 2; 8, 6, 4.
These four cases produce 4, 4, 6, and 6 numbers respectively.
There are 4 + 4 + 6 + 6 = 20 such numbers. There are a total of 4×4×3 = 48 three-digit numbers using the digits 0, 2, 4, 6, and 8.
The probability is $P = \frac{20}{4 \times 4 \times 3} = \frac{5}{12}$.

2. Solution: 9.

A network of the cube is shown below.



A network can be traversed without retracing any edge if it has two odd nodes. A node is one of the 8 vertices. An odd node is a vertex that has an odd number of edges coming out of it. This network has 8 odd nodes, so six such nodes need to be removed. One example is shown. The two odd nodes left with one being the starting point and the other one being the ending point. The longest distance is then 9.

3. Solution: $\frac{11}{18}$.

Let the event of removing a red ball be A_1, the event of removing a white ball be A_2, and the event of selecting a red ball be B. We have

$P(A_1) = \frac{4}{6} = \frac{2}{3}$

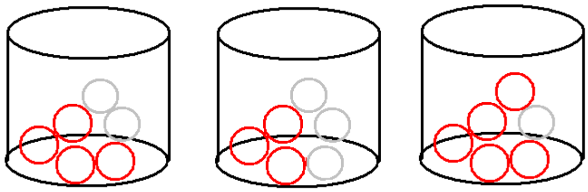
$P(B|A_1) = \frac{3}{6} = \frac{1}{2}$ (The probability of selecting a red ball after removing and replacing a red ball.)

$P(A_2) = \frac{2}{6} = \frac{1}{3}$

$P(B|A_2) = \frac{5}{6}$ (The probability of selecting a red ball after removing and replacing a white ball.)

$$P(B) = P(A_1) \times P(B \mid A_1) + P(A_2) \times P(B \mid A_2) = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{5}{6} = \frac{1}{3} + \frac{5}{18} = \frac{11}{18}.$$

1999 Mathcounts National Sprint Round Solutions



(a) (b) (c)

4. Solution: 56.

Let g be the number of girls and b be the number of boys.

According to the problem, after one-fifth of the girls left, the ratio of the girls to boys was 2:5, or

$$\frac{\frac{4}{5}g}{b} = \frac{2}{3} \quad (1) \qquad \Rightarrow \qquad \frac{4}{5}g = \frac{2}{3}b \quad (2)$$

After 44 boys leave, the ratio of boys to girls was 2:5, or

$$\frac{\frac{4}{5}g}{b-44} = \frac{5}{2} \quad (3)$$

Substituting (2) to (3): $\frac{\frac{2}{3}b}{b-44} = \frac{5}{2}$ $\Rightarrow$(cross multiply) $5(b-44) = \frac{4}{3}b$

$\Rightarrow$ $5b - 220 = \frac{4}{3}b$ $\Rightarrow$ $5b - \frac{4}{3}b = 220$ $\Rightarrow$ $\frac{11}{3}b = 220$ $\Rightarrow$ $b = 60$.

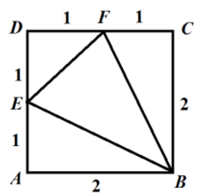
Substitute the value of b into any of the above equations to obtain g = 50.

The number of students who remained $= (b-44) + \frac{4}{5}g = (60-44) + \frac{4}{5} \times 50 = 16 + 40 = 56$.

5. Solution: $\frac{3}{2}$.

The area of ΔBEF = The area of the square $ABCD$ – The area of ΔEDF – The area of ΔBAE – The areas of ΔBCF

$= 2^2 - \frac{1\times 1}{2} - 2\times\frac{1\times 2}{2} = 4 - \frac{1}{2} - 2 = \frac{3}{2}$.

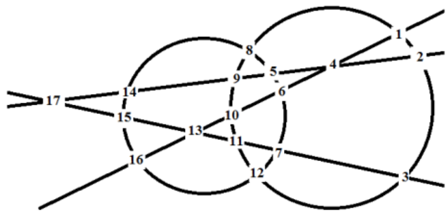


6. Solution: $\frac{1}{5}$.

There are a total of 15 cards. There are $\binom{15}{3}$ ways to choose 3 cards from 15. There are $\binom{8}{3}$ ways to choose 3 red cards from 8 and $\binom{7}{3}$ ways to choose 3 black cards from 7. The probability is equal to $\dfrac{\binom{8}{3}+\binom{7}{3}}{\binom{15}{3}}=\dfrac{1}{5}$.

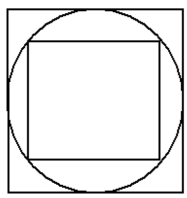
7. Solution: 17.
Method 1:
Three lines can have 3 points of intersection. One circle will have at most 2 points of intersection with each line. So, each circle will add 6 more points and 2 circles will add 12 points of intersection with the lines. However, two circles will also have at most 2 points of intersection with themselves. This will add 2 more points of intersection. The total points of intersection among two circles and three lines are 3 + 12 +2 = 17.

Method 2:
Direct counting:



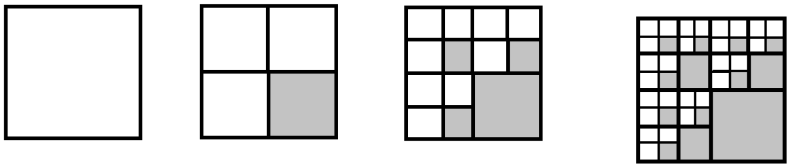
8. Solution: $12\sqrt{15}$.
Using the Heron formula to find the area of the triangle, we get
$s=\frac{1}{2}(a+b+c)=\frac{1}{2}(8+12+16)=18$.
$A=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{18(18-8)(18-12)(18-16)}=12\sqrt{15}$.

9. Solution: $81\sqrt{3}$.
Shown in the 2-*D* image below, the diameter of the sphere is the same as the side length of the large cube. The diameter of the sphere is also the same as the length of the small cube's diagonal. Let the side of the small cube be *a*. It's diagonal has a length of 9, so we know that $a^2+a^2+a^2=9^2$. $a=\frac{9}{\sqrt{3}}$ and the volume of the inscribed cube is



$a^3 = (\frac{9}{\sqrt{3}})^3 = 81\sqrt{3}$.

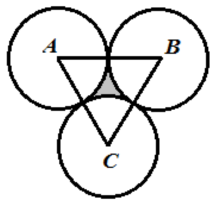
10. Solution: $\frac{37}{64}$.



Stage 0 Stage 1 Stage 2 Stage 3

The nine white squares in stage 2, after partially shaded, will produce $\frac{1}{4} \times 9$ more shaded areas. The fraction of shaded area in the third stage will be $\frac{7+\frac{9}{4}}{16} = \frac{37}{64}$.
Note: See a similar problem in 1995 National Sprint problem 20.

11. Solution: 0.16.



The area of ΔABC equals $\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4} \times 2^2 = \sqrt{3}$. The three white regions represent three pieces of one-sixth of a circle.
The area of the shaded region
= The area of the triangle $ABC - 3 \times \frac{1}{6}$ (the area of the circle of radius 1)
= $\sqrt{3} - \frac{3}{6}\pi \times 1^2 = \sqrt{3} - \frac{\pi}{2} = 1.737 - \frac{3.14}{2}$ =0.16.

12. Solution: $-\frac{1}{8}$.
Method 1:
Square both sides:
$(5x-1)^2 = (3x+2)^2$
$\Rightarrow 25x^2 - 10x + 1 = 9x^2 + 12x + 4$

1999 Mathcounts National Sprint Round Solutions

$16x^2 - 22x - 3 = 0$. Using the quadratic formula, we get $x_1 = \frac{3}{2}$ and $x_2 = -\frac{1}{8}$. The smallest value is $x_2 = -\frac{1}{8}$.

Method 2:

We write the equation as two equations: $5x - 1 = 3x + 2$ (1)

and $5x - 1 = -(3x + 2)$ (2)

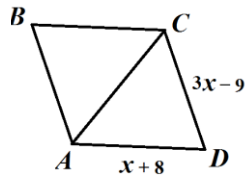
Solve for *x* in (1), we get: $x = \frac{3}{2}$

Solve for *x* in (2), we get: $x = -\frac{1}{8}$. The desired solution is then $x = -\frac{1}{8}$.

13. Solution: 2.

$y = f(x) = \frac{4x+1}{3}$; $x = \frac{3y-1}{4}$; $f^{-1}(1) = \frac{3\times1-1}{4} = \frac{1}{2}$; $(f^{-1}(1))^{-1} = (\frac{1}{2})^{-1} = 2$.

14. Solution: 28.



Using the Triangle Inequality, we have:

$x + 8 + x + 8 > 3x - 9$ (1)

$x + 8 + 3x - 9 > x + 8$ (2)

Solve *x* in (1), we have $25 > x$.

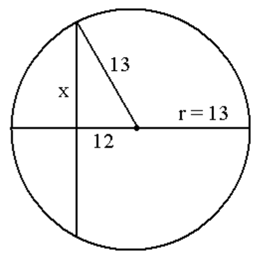
Solve *x* in (2), we have $x > 3$.

The smallest value for *x* is 24 and the largest value is 4. The answer is 24 + 4 = 28.

15. Solution: 10.

The radius of the circle is 26/2 = 13. The radius of the circle is a perpendicular bisector of the chord. Using the Pythagorean Theorem, the desired solution is

$2x = 2 \times \sqrt{13^2 - 12^2} = 2 \times 5 = 10.$



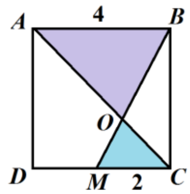
Note: Some Common Pythagorean Triples:

3	4	5
5	12	13
8	15	17
7	24	25
20	21	29
12	35	37
9	40	41
11	60	61
13	84	85

16. Solution: 402.
Method 1:
$12 = 1 \times 2 \times 6 = 1 \times 3 \times 4 = 2 \times 2 \times 3 = 2 \times 6 = 3 \times 4$.
The 2-digit positive integers have the form of $2^{(a+1)} \times 3^{(b+1)} \times 5^{(c+1)} \times 7^{(d+1)} \cdots$
We found the following 2-digit positive integers that have 12 factors:
$2^5 \times 3^1 \times 5^0 = 96$
$2^3 \times 3^2 \times 5^0 = 72$
$2^1 \times 3^2 \times 5^1 = 90$
$2^2 \times 3^1 \times 5^1 = 60$
$2^2 \times 3^1 \times 7^1 = 84$
The desired solution is 60 + 72 +84 + 90 +96 = 402.

Note:
Table of factors of counting numbers 1 to 100

Number of factors	Counting numbers	Number of counting numbers	Property
1	1	1	Square number
2	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97	25	Prime number
3	4, 9, 25, 49	4	Square of a prime
4	6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95	32	
5	16, 81	2	Square number
6	12, 18, 20, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 92, 98, 99	16	
7	64	1	Square number
8	24, 30, 40, 42, 54, 56, 66, 70, 78, 88	10	
9	36, 100	2	Square number
10	48, 80	2	
11		0	
12	60, 72, 84, 90, 96	5	



17. Solution: $\frac{1}{2}$.

$\Delta AOB \sim \Delta COM$ ($AB \| MC$). M is the midpoint of CD. $\frac{OC}{OA}=\frac{2}{4}=\frac{1}{2}$.

18. Solution: $\frac{1}{10}$.

There are $\underline{9}\times\underline{10}\times\underline{1}=90$ 3-digit palindromes. There are $\underline{9}\times\underline{10}\times\underline{10}=900$ 3-digit palindromes. The probability is then $P=\frac{90}{900}=\frac{1}{10}$.

19. Solution: 16.
Let the two numbers be x and y.
According to the problem,
$xy-(x+y)=39 \quad \Rightarrow \quad (x-1)(y-1)=40=10\times 4=8\times 5$.
(Note that 40 = $20\times 2=40\times 1$ are both out of range, since each number must be less than 20.)
$(x-1)=8$ and $(y-1)=5$ gives x = 9 and y = 6. They share 3 as a common factor, so they are not relatively prime.
$(x-1)=10$ and $(y-1)=4$ gives x = 11 and y = 5. The sum is 16.

20. Solution: 38.
$$\frac{9a-8}{10a-8}=\frac{5}{6} \quad \Rightarrow \quad 54a-48=50a-40 \quad \Rightarrow \quad a=2$$
The sum of the original fraction's numerator and denominator is $(9+10)a=38$.

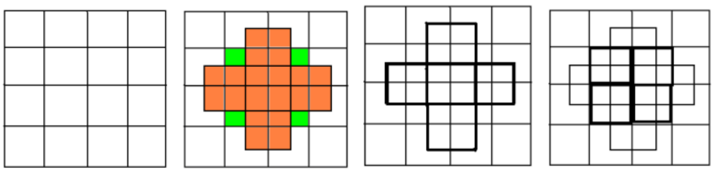
21. Solution: $\frac{32}{65}$.

The total number of outcomes is equal to $\underline{15}\times\underline{14}\times\underline{13}\times\underline{12}$, since the balls are selected without replacement.
We are given 8 odd numbers (1, 3, 5, 7, 9, 11, 13, 15) and 7 even numbers (2, 4, 6, 8, 10, 12, 14).
In order to get an odd sum, we need to have the following selections of the balls:
Case I: Odd, Even, Even, Even.
There are $\underline{8}\times\underline{7}\times\underline{6}\times\underline{5}\times\frac{4!}{3!\times 1!}$ ways in this case.
Case II: Odd, Odd, Odd, Even.

There are $\underline{8}\times\underline{7}\times\underline{6}\times\underline{7}\times\frac{4!}{3!\times 1!}$ ways in this case.

The desired probability is $P=\dfrac{8\times7\times6\times5\times\frac{4!}{3!\times1!}+8\times7\times6\times7\times\frac{4!}{3!\times1!}}{15\times14\times13\times12}=\dfrac{32}{65}$.

22. Solution: 67.
(1) The number of squares in the figure (a): $4^2 + 3^2 + 2^2 + 1^2 = 30$;
(2) The number of shaded squares consisting of one small square: 24;
(3). The number of squares consisting of 4 small squares in (c): 5 + 5 – 1 = 9;
(4) The number of squares in (d): 4.
The total number of squares is 30 + 24 + 9 + 4 = 67.



(a) (b (c) (d)

23. Solution: 4.
Method 1:
When Rachel divides her favorite number by 7, she gets a remainder of 5. The smallest value for her favorite number can be 7 + 5 = 12. 12 × 5 = 60. The remainder will be 4 when 60 is divided by 7.

Method 2:
Let Rachel's favorite number be x.
Since she gets a remainder of 5 after dividing her number by 7, we have
$x = 5 \mod 7$
$5x \equiv 5\times5 \equiv 4 \mod 7$. So the remainder will be 4.

24. Solution: $\frac{13}{20}$.

Let the number picked be x. According to the Triangle Inequality, we have

$$12+7>x \quad\Rightarrow\quad x<19$$
$$7+x>12 \quad\Rightarrow\quad x>5$$

There are a total of 20 numbers and the favorable outcomes are from 6 to 18. The probability is $P = \dfrac{18-6+1}{20} = \dfrac{13}{20}$.

25. Solution: $\dfrac{1}{3}$.

In order for the five-digit number to be divisible by 36, the number needs to be divisible by both 4 and 9. This means that the number formed by the last two digits must be divisible by 4 (the units digit must then be 2 or 6) and the sum of the digit must be divisible by 9.

We then have:

$2 + 1 + x + 7 + 2 \equiv 0 \mod 9 \quad \Rightarrow \quad 3 + x \equiv 0 \mod 9 \quad \Rightarrow \quad x = 6$

$2 + 1 + x + 7 + 6 \equiv 0 \mod 9 \quad \Rightarrow \quad x + 7 \equiv 0 \mod 9 \quad \Rightarrow \quad x = 2$

The ratio of the smaller digit to the larger digit is $\dfrac{2}{6} = \dfrac{1}{3}$.

26. Solution: 5.

Let the first term of the sequence be a. The sum of first ten terms of an arithmetic sequence with a common difference of, say, 1.

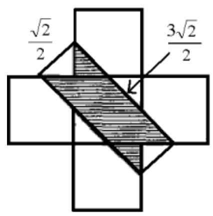
$S = a + a + 1 + a + 2 + .. + a + 10 = 10a + 45 = 5(2a + 9)$.

S must be divisible by 5.

27. Solution: $\dfrac{5}{4}$.

Shown in the figure below, the area of the shaded region is equal to the areas of the two small triangles subtracted from the area of the surrounding rectangle, or

$$\frac{\sqrt{2}}{2} \times \frac{3\sqrt{2}}{2} - \frac{\frac{1}{2} \times \frac{1}{2}}{2} \times 2 = \frac{5}{4}.$$



28. Solution: $\dfrac{1}{4}$.

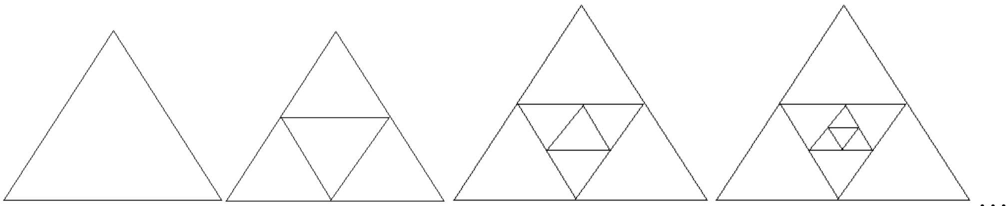
$$f(x) = \frac{1}{1 - \frac{1}{1 - \frac{1}{1-x}}} = \frac{1}{1 - \frac{1}{\frac{1-x}{1-x} - \frac{1}{1-x}}} = \frac{1}{1 - \frac{1}{\frac{-x}{1-x}}} = \frac{1}{1 + \frac{1-x}{x}} = \frac{1}{\frac{x}{x} - \frac{1-x}{x}} = x.$$

1999 Mathcounts National Sprint Round Solutions

$f(-2) = -2$; $f(f(-2)) = f(-2) = -2$; $(f(f(-2)))^{-2} = (-2)^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

29. Solution: $\frac{1}{128}$.

The perimeters form a pattern:



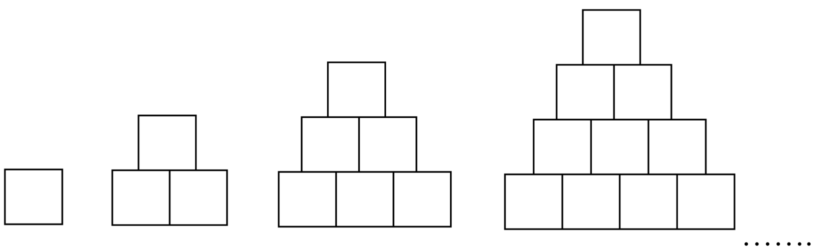
$\frac{3}{2^0}$ $\frac{3}{2^1}$ $\frac{3}{2^2}$ $\frac{3}{2^3}$ … $\frac{3}{2^{n-1}}$

The pattern is a geometric sequence with a common ratio of $\frac{1}{2}$. The ratio of the perimeter of the tenth triangle to the perimeter of the third triangle is

$$\frac{\frac{3}{2^9}}{\frac{3}{2^2}} = \frac{2^2}{2^9} = \frac{1}{2^7} = \frac{1}{128}.$$

30. Solution: 80.

We first find the pattern of the perimeter of the arrangement as shown below, where N is the number of rows in the figure:



2×2 2×4 2×6 2×8 ….. 2×(2n)

The perimeter of the figure with n rows is equal to 2 × (2n).

Next, we find how many rows are in the 20th figure.

$$1 + 2 + 3 + 4 + ... + n = \frac{(1+n)n}{2} = 210 \quad \Rightarrow \quad n = 20.$$

The desired perimeter is 2 × (2n) = 2 × 2 × 20 = 80.