Example 1: A penny $A$ is rolling around a second penny $B$ without slipping until it returns to its starting point. How many revolutions does penny $A$ make?

Solution: Two revolutions.

The distance $D$ traveled by the center of the circle $A$ can be used as a representative distance traveled by the circle $A$.

$$D = 2\pi (r + r) = 4\pi r.$$  

The number of revolutions is $$\frac{4\pi r}{2\pi r} = 2.$$  

Theorem 1: Circle $A$ is rolling around a second circle $B$ without slipping until it returns to its starting point. The number of revolutions the circle $A$ make is

$$N = \frac{2\pi (R + r)}{2\pi r} = \frac{R}{r} + 1.$$  

The total distance the center of the circle travelled is $D = 2\pi (R + r)$.

The distance the center of the circle travelled when circle $B$ travels one revolution is $d = \frac{2\pi r}{R} \times (R + r)$.

Example 2: The side of equilateral $\triangle ABC$ has length $2\pi$. A circle with radius 1 rolls around the outside of $\triangle ABC$. When the circle first returns to its original position, how many revolutions does it roll?

Solution:

The circle rolls a $180^\circ - 60^\circ = 120^\circ$ arc from point $A$ of $AC$ to point $A$ on $AB$ (1/3 of the circumference of the circle). The circle rolls one revolution from point $A$ of $AB$ to point $B$ on $AB$.

Total it rolls 4 revolutions.
**Theorem 2:** Circle $A$ is rolling around a regular $n$ sides polygon with the side length the same as the circumference of the circle without slipping until it returns to its starting point. The number of revolutions the circle $A$ make is $N = n + 1$.

**Theorem 3.1:** Circle $A$ is rolling around a convex polygon without slipping until it returns to its starting point. If the length of the perimeter of the polygon is $n$ times of the length of the circumference of the circle, the number of revolutions the circle $A$ make is $N = n + 1$.

The circle rolls a $(180^\circ - \angle A_1) + (180^\circ - \angle A_2) + \ldots + (180^\circ - \angle A_n)$

$= 180^\circ \times n - (\angle A_1 + \angle A_2 + \ldots + \angle A_n) = 180^\circ \times n - 180^\circ \times (n - 2)$

$= 360^\circ$ arc along all vertices.

**Theorem 3.2:** The distance the center of the circle travels is the sum of the length of the perimeter of the polygon and the length of the circumference of the circle.

**Example 3:** Circle $A$ of radius 1 is rolling around inside a second circle $B$ of radius 4 without slipping until it returns to its starting point. Find the number of revolutions the circle $A$ makes.

Solution:

The distance $D$ traveled by the centre of the small can be used as a representative distance traveled by it.

$$D = 2\pi (R - r) = 2\pi (4 - 1) = 6\pi$$

The number of revolutions is $\frac{6\pi}{2\pi \times 1} = 3$

**Theorem 4:** Circle $A$ with radius $r$ is rolling around inside a second circle $B$ with radius $R$ without slipping until it returns to its starting point. The number of revolutions the circle $A$ makes is $N = \frac{2\pi (R - r)}{2\pi \times r} = \frac{R}{r} - 1$. 

214
**Theorem 5:** A circle with radius \( r \) is rolling around inside a triangle with sides \( a, b, \) and \( c \) without slipping until it returns to its starting point. The distance travelled by the center of the circle is:

The perimeter of \( \triangle ABC \) – the perimeter of the similar triangle \( \triangle DEF = a + b + c - m(a + b + c) = (1 - m)(a + b + c) \)

where \( m = 1 - \frac{pr}{A} \), and \( p = \frac{1}{2}(a + b + c) \)

Below shows how we got \( m \):

We know that \( \triangle DEF \), of the center of the rolling circle, is similar to \( \triangle ABC \), so we label its sides \( ma, mb, mc \), for some \( m \) and \( 1 > m > 0 \).

The area of \( \triangle ABC \) is \( A = \sqrt{p(p-a)(p-b)(p-c)} \)  \hspace{1cm} (1)
where \( p = \frac{1}{2}(a + b + c) \)

The area of \( \triangle DEF \) is \( A_i = \sqrt{mp(mp-ma)(mp-mb)(mp-mc)} = m^2A \)  \hspace{1cm} (2)

Partition \( \triangle ABC \) into three trapezoids of altitude \( r \) and \( \triangle DEF \), and compute the area of \( \triangle ABC \) in terms of \( r \):

\[
[ABC] = [DACE] + [CEFB] + [BFDA] + [DEF] \\
= \frac{1}{2}(r(mb+b)) + \frac{1}{2}(r(ma+a)) + \frac{1}{2}(r(mc+c)) + m^2A \\
= \frac{1}{2}(r(mb+b+ma+a+mc+c)) + A = pr(m+1) + m^2A \]  \hspace{1cm} (3)

We know that (1) = (3). So we have

\[ A = pr(m+1) + m^2A \Rightarrow pr(m+1) + m^2A - A = 0 \]
\[ \Rightarrow pr(m+1) + A(m+1)(m-1) = 0. \]

Since \( m \neq 0, m + 1 \neq 0 \). Thus \( pr + A(m-1) = 0 \)  \hspace{1cm} \( \Rightarrow \)

\[ m = 1 - \frac{pr}{A}. \]
PROBLEMS

Problem 1: A small circle of radius 2 cm is rotating without slipping around a larger circle of radius 10 cm. When the circle first returns to its original position, how many revolutions does the circle roll?

Problem 2: The side of a regular hexagon has length $2\pi$. A circle with radius 1 rolls around the outside of it. When the circle first returns to its original position, how many revolutions does the circle roll?

Problem 3: Circle $A$ with radius 2 is rolling around inside a second circle $B$ with radius 8 without slipping until it returns to its starting point. Find the number of revolutions the circle $A$ makes.

Problem 4: A small circle of radius 2 cm is rotating without slipping around a larger circle of radius 9 cm. If the small circle starts with point $A$ on its circumference in contact with the larger circle, find the exact distance traveled by the centre of the small circle before the point $A$ next comes in contact with the large circle.

Problem 5: (UNL Probe I, 1993) A circle is rolled without slipping, across the top of the other six identical circles to get from the initial position $x$ to the final position $y$. What is the number of revolutions it must make?

Problem 6: (1993 AMC 12) The sides of $\triangle ABC$ have lengths 6, 8 and 10. A circle with center $P$ and radius 1 rolls around the inside of $\triangle ABC$, always remaining tangent to at least one side of the triangle. When $P$ first returns to its original position, through what distance has $P$ traveled?
SOLUTIONS:

Problem 1: Solution:
By Theorem 1, the number of revolutions the circle A make is

\[ N = \frac{2\pi (R + r)}{2\pi r} = \frac{R}{r} + 1 = \frac{10}{2} + 1 = 6. \]

Problem 2: Solution:
By Theorem 2, the number of revolutions the circle A make is \( N = n + 1 = 6 + 1 = 7 \).

Problem 3: Solution:
By Theorem 4, the number of revolutions the circle A makes is \( N = \frac{R}{r} - 1 = \frac{8}{2} - 1 = 4 - 1 = 3 \).

Problem 4: Solution:
Method 1:
Since the circumference of the small circle is \( 4\pi \), the central angle in the large circle between successive points of contact of the point A with the large circle is \( \theta = s_1/r = 4\pi/9 \).

The radius of the circle followed by the centre of the small circle is 11. Thus, the distance traveled by the centre of the small circle before the point A next comes in contact with the large circle is \( s_2 = R\theta = (9 + 2) (4\pi/9) = 44\pi/9 \).

Method 2:
By the Theorem 1, \( d = \frac{2\pi}{R} \times (R + r) = \frac{2\pi \times 2}{9} \times (9 + 2) = \frac{44\pi}{9} \).

Problem 5: Solution:
As a circle rolls through one revolution, its center travels a distance equal to the circumference. As the circle moves from position X to position p₁, the center moves a
distance $2\pi/3$ $(2r) = (4\pi r)/3$. As the circle moves from position $p_1$ to $p_2$, the center moves $\pi/3$ $(2r) = (2\pi r)/3$. The total distance traveled by the center of the rolling circle is $2(4\pi r)/3 + 4(2\pi r)/3 = (16\pi r)/3$. Since each revolution is $2\pi r$, the number of revolutions is $16\pi r/3$ divided by $2\pi r$ which is $8/3$.

**Problem 6:** Solution: 12
We solve this problem by two different methods other than the official solution.

Method 1:

By **Theorem 5**:

The distance $= a + b + c - m(a + b + c) = (1 - m)(a + b + c)$.

\[
m = 1 - \frac{pr}{A} = 1 - \frac{2}{8 \times 6} \times 1 = 1 - \frac{1}{2} = \frac{1}{2}.
\]

So the distance $= (1 - m)(a + b + c) = (a + b + c)/2 = 12$.

Method 2:

We have $\tan \theta = \frac{y}{8}$. By the angle bisector Theorem, we get $\frac{8}{y} = \frac{10}{6 - y} \Rightarrow y = \frac{8}{3}$.

\[
\tan \theta = \frac{r}{m} = \frac{1}{m} \quad \text{so} \quad \frac{1}{m} = \frac{8}{3} \quad \Rightarrow \quad m = 3
\]

\[
\tan \alpha = \frac{r}{n} = \frac{1}{n} \quad \text{and} \quad \frac{1}{n} = \frac{3}{6} \quad \Rightarrow \quad n = 2.
\]

Thus the distance travelled $= 8 - (m + 1) + 6 - (n + 1) + 10 - (m + n) = 8 - (3 + 1) + 6 - (2 + 1) + 10 - (3 + 2) = 12$. 

218