ARML POWER QUESTION – 2006–The Power of Origami

The power question is worth 40 points. Each problem is worth 4 points. To receive full credit the presentation must be legible, orderly, clear, and concise. Even if not proved, earlier numbered items may be used in solutions to later numbered items but not vice-versa. The pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS.

Put the TEAM NUMBER (not the Team name) on the cover sheet used as the first page of the papers submitted. Do not identify the Team in any other way.

We all know how to fold a unit square into a square of area 1/4. What other fractions of a unit square's area can be constructed by folding? We're going to investigate this question and a few others…

For the purposes of this problem, there are six Basic Origami Operations. Each is a fold that creates a crease.

B1: Given points P and Q, you can make a fold creating a crease passing through points P and Q.

B2: Given points P and Q, you can fold P onto Q.

B3: Given lines ℓ_1 and ℓ_2, you can fold ℓ_1 onto ℓ_2.

B4: Given line ℓ and point A which can be on or off line ℓ, you can make a fold creating a crease through point A perpendicular to line ℓ. Note: A can't be an endpoint of a segment.

B5: Given line ℓ, point P not on line ℓ, and point Q not necessarily on line ℓ, you can make a fold that places P onto line ℓ while the crease passes through Q, unless no fold satisfying these conditions exists.

B6: Given points P and Q as well as lines ℓ_1 and ℓ_2, you can make a fold that simultaneously places P onto ℓ_1 and Q onto ℓ_2, unless no fold satisfying these conditions exists.

Note: in the operations above, points P and Q and lines ℓ_1 and ℓ_2 are not necessarily distinct.

To start, we're given a unit square's vertices as points and sides as segments. As with straightedge-and-compass constructions, we assume you can construct an arbitrary but nonspecial point or segment at will: for example, given a segment, you can construct a point P "somewhere" on that segment, but you cannot claim that P is the midpoint, or any other point with special properties, without justifying P's construction using B1–B6; similarly, given point P you can construct a segment L passing through it without further justification so long as you do not assume L has any particular properties (e.g. being perpendicular to a given segment).

1. a) What line is formed as a result of B2?

   b) If ℓ_1 and ℓ_2 intersect at point P, what line is formed as a result of B3?

   c) The statement of B5 is equivalent to intersecting a line with a circle. Describe the circle and the line, and justify the equivalence.

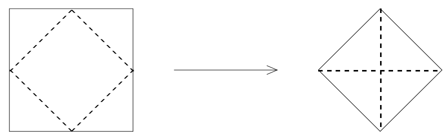
2. a) Show that B3 is a special case of B6.

   b) B5 suggests that for some locations of P, Q, and ℓ, there is no fold that would send P onto ℓ while passing through Q. Give an example of such a situation and explain why no such fold exists.

We're ready to start folding squares. To explain a particular construction, you must describe it in terms of B1–B6 above. Apart from the vertices and edges of a square, you must justify the construction of any other points or segments in terms of B1–B6. In general, you need not justify the claim that B5 or B6 is possible in a given situation.

3. a) Describe a way to fold a unit square $ABCD$ into four squares of area 1/4 in terms of B2–B6. We'll call this the Quarter construction.

   b) In his dialogue *Meno,* Plato describes a way of folding a square into a square of 1/2 the original area by folding each corner into the center as shown below. Unfortunately, that presupposes that the center of the square is already located, which isn't the case. Describe the construction of a square 1/2 the area of the original in terms of B's. We'll call this the Meno construction.



4. Starting with a unit square and using only creases formed by Quarter and/or Meno constructions to form squares, what fractional areas p/q can be obtained where p and q are integers? Prove your result.

5. a) For $n = 1, 2, \ldots, 6$, how many different ways are there to obtain a square of area $1/2^n$ from a unit square using only Quarter and/or Meno constructions? Order matters.

   b) Let $F(n)$ be the number of ways of obtaining a square of area $1/2^n$ using only Quarter and/or Meno constructions. Find and prove a formula, recursive or explicit, for $F(n)$ if order matters.

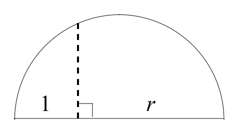
6. One side of a square can be folded into thirds, although it's not as easy as it appears: simply saying "fold it into thirds" begs the question of how the trisection points can be constructed.

   a) Using only B's construct points T_1 and T_2 that trisect one side of the square. Justify.

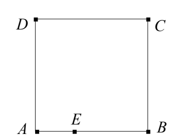
   b) Show how to construct points that divide the side of a square into n equal pieces for any $n > 2$.

7. Describe, with proof, what fractions p/q can be obtained as areas of squares folded from a single unit square using only creases formed by combinations of the constructions in #6, Quarter constructions and Meno constructions, possibly including more than one of each. Your proof does not have to *exclude* other fractions, but must justify the inclusion of every fraction described.

8. a) The diagram shows the standard straightedge-and-compass construction of a segment of length $\sqrt{r}$ (shown dotted in the diagram). Given an infinitely large sheet of paper and segments of length 1 and r, collinear with and adjacent to each other, describe an equivalent construction in terms of B's. You can't use a straightedge or compass.



   b) Given $AE = r < 1$, describe how to construct a segment of length $\sqrt{r}$ on the unit square $ABCD$ using only B's such that no part of the segment lies outside the unit square.



9. Describe, with proof, what fractions p/q can be obtained as areas of squares folded from a single unit square using only creases formed by combinations of the constructions in #6 & #8, Quarter constructions and Meno constructions, possibly including more than one of each. Your proof does not have to *exclude* other fractions, but must justify the inclusion of every fraction described.

10. a) A segment of length $\sqrt{3}/2$ can be constructed using two folds, and so a square of area 3/4 can be constructed in 5 folds including folding down the edges to make an actual square. Show how to do so.

    b) The method developed in #9 requires 12 folds to construct a square of area 1/5 by locating all 4 vertices. Find a construction that requires fewer folds.