ARML POWER QUESTION – 2005 – Cantor County, the home of the Aleph-Naughts

The power question is worth 40 points. Each problem is worth 4 points. To receive full credit the presentation must be legible, orderly, clear, and concise. Even if not proved, earlier numbered items may be used in solutions to later numbered items but not vice-versa. The pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS.

Put the TEAM NUMBER (not the Team name) on the cover sheet used as the first page of the papers submitted. Do not identify the Team in any other way.

In Cantor County the address h of each building is an infinitely long sequence $a_1 a_2 a_3 \ldots$ where each $a_i = 0$ or 1. For example, we'll write $h = 0110100111011\ldots$ to indicate the address of a building. If the address has a repeating block of digits as in $h = 011110011001100\ldots$, we'll indicate that using bar notation drawn from repeating decimals, i.e. $h = 011\overline{1100}$. We will assume that all possible addresses are in use, i.e., each string refers to an actual building.

The Cantor Postal Service measures the distance between two buildings whose addresses are h_1 and h_2 in terms of the addresses. Let $h_1 = a_1 a_2 a_3 \ldots$, $h_2 = b_1 b_2 b_3 \ldots$, and let $D = \{i : a_i \neq b_i\}$. If $D \neq \varnothing$, then the *distance* between h_1 and h_2, written as $d(h_1, h_2)$, equals 2^{-n} where n is the smallest element of D. If $D = \varnothing$, then $d(h_1, h_2) = 0$. The address of the Cantor post office is $0000\ldots = \overline{0}$.

1. Let $h_1 = \overline{110}$ and $h_2 = 10011110111100\ldots$.

   a) Compute $d(h_1, h_2)$.

   b) Determine the addresses of three buildings h such that $d(h_1, h) = \frac{1}{8}$.

   c) Determine the addresses of three buildings h, $h \neq h_1$, such that $d(h_1, h) < \frac{1}{8}$.

   d) Prove that $d(h_m, h_n) = 0$ if and only if $h_m = h_n$.

2. a) Georg leaves the Cantor post office, goes to Cantor High School, address $\overline{001}$, and then to the hospital, address $\overline{0011}$. Compute the difference between the length of that trip and the length of the trip from the post office directly to the hospital.

   b) Prove that d satisfies the triangle inequality. That is, if h_1, h_2, and h_3 are any three distinct addresses, then $d(h_1, h_2) + d(h_2, h_3) \geq d(h_1, h_3)$.

   c) Prove that d never satisfies the Pythagorean Theorem. That is, prove there are no distinct addresses h_1, h_2, and h_3 such that $\left(d(h_1, h_2)\right)^2 + \left(d(h_2, h_3)\right)^2 = \left(d(h_1, h_3)\right)^2$.

3. A postal route starts at the post office and delivers to an infinite sequence of buildings $h_1, h_2, h_3, \ldots$. A postal route is called *desirable* if $d(h_1, h_2) > d(h_2, h_3) > d(h_3, h_4) > \ldots$.

   a) Determine a postal route whose total length is 1.

   b) Prove that any desirable route has a finite total length.

The post office ordered a sorting machine A that turned out to be defective. It *deletes* the first (leftmost) three digits of a building's address and the mail is then sent to the new address. For example, the Smiths' address is $1100\overline{1100}$, but their mail is now sent to a building whose address is $01100\overline{1100}$.

4. a) The Smiths now receive several buildings' mail. Give the address of one of those buildings.

   b) How many buildings' mail do the Smiths now receive? Justify your answer.

5. a) The Smiths complain to their alderman, only to discover that *his* house still receives its own mail. Demonstrate how that is possible.

   b) Determine, with proof, the number of buildings that receive their own mail under these circumstances.

6. The postal service discovers that all machines from this manufacturer delete a *consecutive string* of digits of finite length from the beginning of an address. Determine those buildings that would *never* receive their own mail, no matter how many digits are deleted.

Disgusted, the postal service composts A and orders a sorting machine B from a different manufacturer. But this machine *inserts* the digits 1001 at the beginning of every address and the mail is delivered to the new address.

7. a) Using B, is it possible that a building could get mail originally addressed to more than one other building? Justify your answer.

   b) Again, it turns out that *every* machine from the second manufacturer has a unique string of digits of finite length that it inserts at the start of an address. Prove that no matter which of this manufacturer's machines is used, there is always a building that still receives its own mail.

A neighborhood N of size $r > 0$ around h_1 consists of a set of addresses h whose distance from h_1 is less than r. It is defined by $N(h_1, r) = \{h : d(h, h_1) < r\}$.

8. a) If $h = \overline{1100}$, list three elements h_i of $N\left(h, \frac{1}{16}\right)$ and describe all such h_i in general.

   b) Is it possible to describe the set of all addresses at a distance of *more* than 1/16 from the Cantor post office as a neighborhood of size $r < 1/2$ around any particular address in Cantor County? If so, give the address and the value of r. If not, demonstrate why not.

9. Prove that if h_1 and h_2 are distinct addresses in Cantor County, then there exist neighborhoods $N(h_1, r_1)$ and $N(h_2, r_2)$ such that $N(h_1, r_1) \cap N(h_2, r_2) = \varnothing$.

10. The post office decides to use the digit 2 as well. Each existing building is assigned a new address whose digits are chosen from $\{0, 1, 2\}$ instead of $\{0, 1\}$. As in the old address system, a digit need not be used. However, when the post office starts assigning addresses to *new* buildings, the bureaucrats realize, to their shock, that the way in which the post office assigned new addresses to old buildings used up all the new addresses. Show that this is possible by constructing a 1-to-1 correspondence between the old addresses and the new addresses.